WAVE EQUATION

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < +\infty \qquad t>0 \qquad c>0$$

note that this eqn in not canonical for

$a_{11} = 1 \qquad a_{12} = 0 \qquad a_{22} = -c^2 \quad \Rightarrow \quad a_{12}^2 - a_{11}a_{22} = c^2 > 0$ hyperbolic eqn

transform it to canonical form

$$\frac{dx}{dt} = \frac{a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \frac{dt}{dx} = \pm c \quad \Rightarrow \quad \begin{array}{ll} dx = c\,dt & dx = -c\,dt \\ x = ct + c_1 & x = -ct + c_2 \\ \xi = x - ct & \eta = x + ct \end{array}$$

use ξ, η and find u_x, u_y, u_{xx}, u_{yy}

$u_x = u_\xi \xi_x + u_\eta \eta_x = c u_\xi + c u_\eta$

$u_t = u_\xi \xi_t + u_\eta \eta_t = u_\xi + u_\eta$

$u_{xx} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$

$u_{tt} = \cdots$ then we get $u_{\xi\eta} = 0$

20 Mart Perş

we got $u_{\xi\eta} = 0$. We can solve it.

$$(u_\xi)_\eta = 0 \Rightarrow u_\xi = P(\xi) \Rightarrow u = \underbrace{\int P(\xi)\,d\xi}_{F(\xi)} + G(\eta)$$

so gen soln : $F(\xi) + G(\eta)$

hence $u_{tt} - c^2 u_{xx} = 0$ has gen soln $u = F(x+ct) + G(x-ct)$

Cauchy Problem (D'elambert)

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < +\infty \qquad t>0$$

$u(x,0) = f(x) \qquad -\infty < x < +\infty$
$u_t(x,0) = g(x) \qquad -\infty < x < +$ $\quad$ (initial cond $t=0$)

let us solve this problem

we know that $u = F(x+ct) + G(x-ct)$ now find F, G

$u(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x)$

$u_t(x,0) = g(x) \Rightarrow cF'(x+ct) - c\,G'(x-ct)\Big|_{t=0} = g(x)$

so we have

$$\begin{cases} F(x) + G(x) = f(x) \\ F'(x) - G'(x) = \frac{1}{c} g(x) \end{cases} \overset{\text{integrate}}{\Longrightarrow} \begin{cases} F(x) + G(x) = f(x) \\ F(x) - G(x) = \frac{1}{c}\int_0^x g(\tau)\,d\tau + A \end{cases}$$

$$\begin{cases} F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\tau)\,d\tau + A/2 \\ G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\tau)\,d\tau - A/2 \end{cases}$$

now, $u(x,t) = F(x+ct) - G(x-ct)$

d'Alembert Formula : $$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} g(\tau)\,d\tau - \int_0^{x+ct} g(\tau)\,d\tau\right)$$

ex Solve the problem

$u_{tt} - 4u_{xx} = 0 \quad -\infty < x < \infty, \quad t > 0 \quad (c^2 = 4 \quad c > 0)$

$u(x,0) = \sin x \quad -\infty < x < \infty$

$u_t(x,0) = x \quad -\infty < x < \infty$

use d'Alembert formula with $f(x) = \sin x$ $g(x) = x$

$$u(x,t) = \frac{\sin(x+2t) + \sin(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \tau\,d\tau$$

$$= \frac{\sin(x+2t) + \sin(x-2t)}{2} + \frac{1}{8}\tau^2\Big|_{x-2t}^{x+2t}$$

$$= \frac{\sin(x+2t) + \sin(x-2t)}{2} + \frac{(x+2t)^2 - (x-2t)^2}{8} = \sin x . \cos 2t + xt$$

ex find general soln of

$$u_{tt} - 3u_{xt} + 2u_{xx} = 0 \qquad -\infty < x < \infty \qquad t > 0$$

type? $a_{11} = 1$ $\quad a_{12} = -3/2$ $\quad a_{22} = 2$ $\qquad a_{12}^2 - a_{11}a_{22} = \frac{9}{4} - 2 = \frac{1}{4} > 0$ hyperbolic type

$$\frac{dx}{dt} = \frac{a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \frac{\frac{-3}{2} \mp \sqrt{\frac{1}{4}}}{1} = -\frac{3}{2} + \frac{1}{2} = -1 \Rightarrow x + t = c_1 = \xi$$

$$-\frac{3}{2} - \frac{1}{2} = -2 \Rightarrow x + 2t = c_2 = \eta$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + 2u_\eta$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi} + 3u_{\xi\eta} + 2u_{\eta\eta}$$

$$u_{tt} = u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta}$$ Now subst into the eqn

$$\underbrace{[u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta}]}_{u_{tt}} - 3\underbrace{[u_{\xi\xi} + 3u_{\xi\eta} + 2u_{\eta\eta}]}_{u_{xt}} + 2\underbrace{[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}]}_{u_{xx}} = 0$$

now we have $u_{\xi\eta} = 0$ $\quad$ we know $u = F(\xi) + G(\eta)$ on (x,t) var.

If I had $-u_{\xi\eta} = 0$
does it change the soln?

$$\boxed{u = F(x+t) + G(x+2t)}$$

Application of d'Alembert formula ex find all solns of

$$\underbrace{u_{ttx}}_{(u_x)_{tt}} - u_{xxx} = 0 \qquad -\infty < x < \infty \qquad t > 0$$

$$u_x(x,0) = 0$$

$$(u_x)_t(x,0) = e^x$$

$v = u_x$ then for v $\Rightarrow$ $\left.\begin{array}{l} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = e^x \end{array}\right\}$ use d'Alembert

$$v = \frac{1}{2}\int_{x-t}^{x+t} e^\tau d\tau \qquad (f = 0 \quad g = e^x) \Rightarrow v = \frac{e^{x+t} - e^{x-t}}{2}$$ but we said $v = u_x$

$u_x = \dfrac{e^{x+t} - e^{x-t}}{2} \overset{\text{integrate}}{\Longrightarrow} u = \displaystyle\int \dfrac{e^{x+t} - e^{x-t}}{2}\, dx$ when to use indef integral ?

$u = \dfrac{e^{x+t}}{2} - \dfrac{e^{x-t}}{2} + A(t)$

Solving the prev ex again

find all solns of

$u_{ttx} - u_{xxx} = 0 \quad -\infty < x < \infty \quad t > 0$

$(u_x)_{tt}$

$u_x(x,0) = 0$

$(u_x)_t(x,0) = e^x$

lowering the order by 1 derivative

$v = u_x$ then for $v \Rightarrow$

$$\left.\begin{aligned} v_{tt} - v_{xx} &= 0 \\ u_x(x,0) = v(x,0) &= 0 \\ u_{xt}(x,0) = v_t(x,0) &= e^x \end{aligned}\right\} \text{use dAlembert}$$

for the 1D wave eqn with wave speed $c=1$, d'Alembert soln :

$$v(x,t) = \frac{1}{2}\left[v(x+t,0) + v(x-t,0)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_t(\xi,0)\, d\xi$$

because $v(x,0) = 0$, first term vanishes

Hence, $v(x,t) = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} e^z\, dz$

$$= \frac{1}{2}\left[e^{x+t} - e^{x-t}\right] = e^x\left(\frac{e^t - e^{-t}}{2}\right) = e^x \sinh t$$

So we have $v(x,t) = u_x(x,t) = e^x \sinh t$

now integrate v wrt x to recover u

$u(x,t) = \displaystyle\int v(x,t)\,dx = \int e^x \sinh t\, dx = e^x \sinh t + A(t)$ where $A(t)$ is arbitrary fn of "t"

the given data $u_x(x,0) = 0$ already satisfied for every $A(t)$ because A is x- independent

The PDE itself requires $u_{tt} - u_{xx} = A''(t) = 0 \Rightarrow A(t) = at + b \quad a,b \in \mathbb{R}$

If we impose $u(x,0) = 0$ then $b = 0$, if no further cond given set $a = 0$

$\Rightarrow u(x,t) = e^x \sinh t$

Thm (Cauchy Problem for Wave Eqn)

if $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ then the Cauchy Problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < +\infty, \quad t > 0$$
$$u(x,0) = f(x) \qquad -\infty < x < +\infty$$
$$u_t(x,0) = g(x) \qquad -\infty < x < +\infty$$

is well posed in $\Omega_T = \{(x,t) \mid -\infty < x < +\infty, 0 < t < T\}$

Pf

1) **Existence** follows from d'Alembert formula

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau$$

2) **Uniqueness** Note that $F(x+ct)$ and $G(x-ct)$ from the general soln are uniquely determined by $f(x)$ and $g(x)$

3) **Stability** (in supremum norm) Consider two solns u_1, u_2 st

$$u_1(x,0) = f_1(x) \qquad u_2(x,0) = f_2(x)$$
$$\frac{\partial}{\partial t} u_1(x,0) = g_1(x) \qquad \frac{\partial}{\partial t} u_2(x,0) = g_2(x)$$

assume $|f_1(x) - f_2(x)| < \delta$ and $|g_1(x) - g_2(x)| < \delta \qquad -\infty < x < \infty$

now estimate $|u_1(x,t) - u_2(x,t)|$

$$|u_1 - u_2| = \left| \frac{f_1(x+ct) + f_1(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(\tau)\, d\tau - \frac{f_2(x+ct) + f_2(x-ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} g_2(\tau)\, d\tau \right|$$

$$|u_1 - u_2| < \left| \frac{f_1(x+ct) - f_2(x-ct)}{2} \right| + \left| \frac{f_1(x+ct) - f_2(x-ct)}{2} \right| + \frac{1}{2c} \left| \int_{x-ct}^{x+ct} g_1(\tau)\, d\tau - \int_{x-ct}^{x+ct} g_2(\tau)\, d\tau \right|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \underbrace{|g_1(\tau) - g_2(\tau)|}_{\leq \delta}\, d\tau$$

$$|u_1 - u_2| \leq \delta + \frac{1}{2c}\,\delta\, |(x+ct) - (x-ct)| \leq \delta + \delta t$$

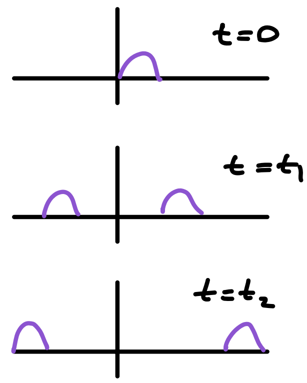
Since $0 \leq t \leq T \Rightarrow |u_1 - u_2| < \delta(1+T)$. Given any $\varepsilon > 0$ take $\delta = \frac{\varepsilon}{1+T}$ then

$|f_1 - f_2| < \delta$ and $|g_1 - g_2| < \delta$ will imply $|u_1 - u_2| < \varepsilon$. We have cont dependence of soln on the initial cond in Ω_T $\square$

Remark in $\Omega = \{(x,t) : -\infty < x < \infty \ , \ 0 < t < \infty\}$

the soln does NOT continuously depend on the initial conditions

interpretation of soln for simplicity we take



$$\left.\begin{array}{l} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array}\right\} \ u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

in general, $$u(x,t) = \frac{f(x+ct) - f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz + \frac{1}{2} f(x-ct) + \frac{1}{2c}\int_0^{x-ct} g(z)\,dz$$

we see that u is the sum of two waves

$$F(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz \quad : \text{moving left with velocity} = c$$

$$G(x-ct) = \frac{1}{2} f(x-ct) + \frac{1}{2c}\int_0^{x-ct} g(z)\,dz \quad : \text{moving right with velocity} = c$$

8 Nisan Salı

Recall we consider the wave eqn $u_{tt} - c^2 u_{xx}$ $\quad t>0 \quad -\infty < x < \infty$

general soln $u(x,t) = F(x-ct) + G(x+ct)$

we can solve the Cauchy problem

$$\left.\begin{array}{lll} u_{tt} - c^2 u_{xx} = 0 & t>0 & -\infty < x < \infty \\ u(x,0) = f(x) & & -\infty < x < \infty \\ u_t(x,0) = g(x) & & -\infty < x < \infty \end{array}\right\} u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2}\int_{x-ct}^{x+ct} g(z)\,dz$$

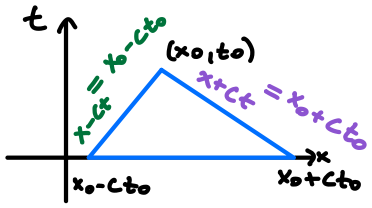
we call the lines $x+ct = a$, $x-ct = b$ the characteristics

Domain of dependence and Domain of influence

Domain of dependence What should we know about the initial data to find $u(x_0,t_0)$?

$$u(x_0,t_0) = \frac{f(x_0-ct_0) + f(x_0+ct_0)}{2} + \frac{1}{2}\int_{x_0-ct_0}^{x_0+ct_0} g(z)\,dz \ .$$ So we need f and g on $[x_0-ct_0 \, , \, x_0+ct_0]$

We call the interval $[x_0 - ct_0, x_0 + ct_0]$ the domain of dependence for (x_0, t_0)



the triangle with vertices (x_0, t_0), $(x_0 - ct_0, 0)$, $(x_0+ct_0, 0)$ is called the characteristic triangle.

ex for the Cauchy problem

$$u_{tt} - 4u_{xx} = 0 \quad \to c=2$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

$\Rightarrow$ its known that f, g are zero on $[2,6]$.

find all pts (x,t) st we can guarantee $u(x,t) = 0$

we need all (x,t) with domain of dependence inside $[2,6]$

$$[x-2t, x+2t] \subset [2,6] \quad \Rightarrow \quad x-2t \geq 2 \qquad x+2t \leq 6$$

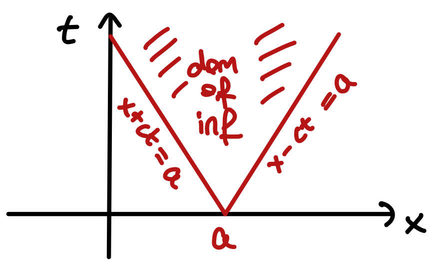
$u(x,t) = 0$ if $(x,t) \in \{(x,t) : x-2t \geq 2$ and $x+2t \leq 6\}$

Domain of influence if we change the initial data in small n-hood of $(a,0)$ at what pts (x,t) the value of $u(x,t)$ changes?

from defn of domain of dependence, we know value of $u(x,t)$ will change if "a" belongs to domain of dependence for pt (x,t)

so, $a \in [x-ct, x+ct]$ or $x - ct \leq a$ AND $x+ct \geq a$

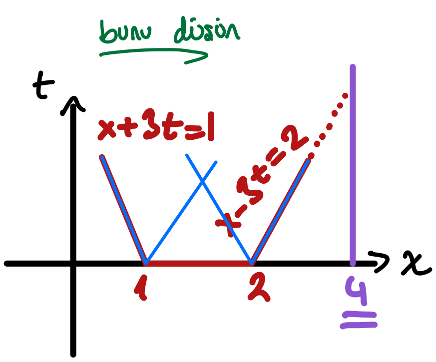
the set $\{(x,t) : x-ct \leq a$ and $x+ct \geq a\}$ is called domain of influence for pt $(a,0)$



ex suppose a wave is described by

$$u_{tt} - 9u_{xx} = 0 \qquad t>0 \qquad -\infty < x < \infty$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

where f, g are zero outside $[1,2]$. At what time the wave will reach pt $x_0 = 4$?



Soln at $t=0$, wave is located on $[1,2]$. we need t_0 for which $(4, t_0)$ belongs to domain of influence for $[1,2]$

$4+3t \geq 1$ always true $(t>0)$

$4-3t \leq 2$

$\Rightarrow 3t_0 \geq 2 \qquad t_0 \geq 2/3$

after $2/3$ units of time wave comes to $x_0=4$

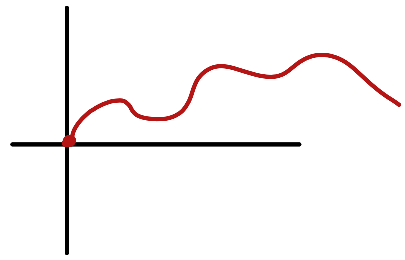
Remark domain of influence of $[1,2]$ is union of dom of inf for $(x,0)$ and $(2,0)$

we need to s.t $\underbrace{4+3t_0 \geq 1}_{\text{always true}}$ and $4-ct_0 \leq 2$ $(c=3)$

$t_0 \geq 2/3$

Semi infinite String

consider the following initial boundary value problem



$$u_{tt} - c^2 u_{xx} = 0 \qquad t>0 \quad x>0$$

fixed end

$$\left.\begin{array}{ll} u(x,0) = f(x) & x>0 \\ u_t(x,0) = g(x) & x>0 \end{array}\right\} \text{initial cond.}$$

$u(0,t) = 0$

$$u(0,t) = 0 \qquad t>0 \; \} \text{ boundary cond.}$$

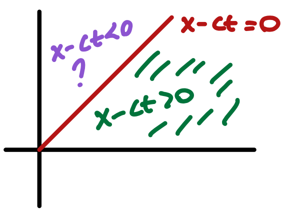
Soln d'Alembert formula

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

BUT there is a problem!

f and g defined for positive values of argument.

if $x-ct<0$ then $f(x-ct)$ and $g(x-ct)$ are not defined!



$$u(x,t) = F(x-ct) + G(x+ct)$$

$$G(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz \quad \checkmark$$

$$F(x-ct) = \frac{f(x-ct)}{2} + \frac{1}{2}\int_0^{x-ct} g(z)\,dz \quad ?$$

$F(x-ct)$ is NOT defined for $x-ct<0$. let us use boundary cond $u(0,t)=0$

$$u(0,t) = 0 \Rightarrow F(-ct) + G(ct) = 0$$

$$F(-ct) = -G(ct) \qquad \text{set } z=-ct \qquad \text{well defined } z<0$$

$$F(z) = -G(\underbrace{-z}_{>0})$$

so, if $x-ct<0$ $\quad F(x-ct) = -G(-(x-ct)) = -G(ct-x)$

$$= -\left[\frac{f(ct-x)}{2} + \frac{1}{2c}\int_0^{ct-x} g(z)\,dz\right.$$

the soln for semi infinite string but fixed end

$$u(x,t) = \begin{cases} F(x-ct) + G(x+ct) = \dfrac{f(x+ct) + f(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\tau)\,d\tau, & \underline{x-ct \geq 0} \\ \underbrace{F(x-ct)}_{-G(ct-x)} + G(x+ct) = \dfrac{f(x+ct) - f(ct-x)}{2} + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(\tau)\,d\tau, & \underline{x-ct < 0} \end{cases}$$

bdry dirsn

ex Solve the problem and evaluate u at $(1,2)$ and $(2,1)$

$$u_{tt} - u_{xx} = 0 \qquad t>0 \quad x>0$$

$$\left.\begin{aligned} u(x,0) &= x^2 \qquad x>0 \\ u_t(x,0) &= 6x \qquad x>0 \end{aligned}\right\} \text{Initial cond}$$

$$u(0,t) = 0 \qquad t>0 \;\Big\}\; \text{boundary cond}$$

soln $c=1$

$$u(x,t) = \begin{cases} \dfrac{(x-t)^2 + (x+t)^2}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} 6\tau\,d\tau & x \geq t \\ \dfrac{-(t-x)^2 + (x+t)^2}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} 6\tau\,d\tau & x<t \end{cases}$$

$$u(x,t) = \begin{cases} x^2+t^2+6xt, & x \geq t \\ 8xt, & x<t \end{cases}$$

$\underset{1<2}{u(1,2)} = 8.1.2 = 16$

$\underset{2>1}{u(2,1)} = 2^2+1^2+6.2.1 = 17$

Remark one can show if f and g are odd fns in general Cauchy problem then $u(x,t)$ is also odd fn wrt x $\quad u(x,t_0) = -u(-x,t_0) \quad \forall t_0$

So $u(0,t_0) = 0 \quad \forall t_0$. Hence, we can solve the initial boundary problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad t>0 \quad x>0$$

? $u(x,0) = f(x) \qquad x>0$

? $u_t(x,0) = g(x) \qquad x>0$

$u(0,t) = 0$

as follows f and g are odd fns to $(-\infty,\infty)$

let $\tilde{f}$ and $\tilde{g}$ be the extended then Solve

$$u_{tt} - c^2 u_{xx} = 0$$

ask $\quad u(x,0) = \tilde{f}(x) \qquad -\infty<x<\infty$

$u_t(x,0) = \tilde{g}(x) \qquad -\infty<x<\infty$

The solution will solve the above problem

What is $\tilde{f}$, $\tilde{g}$?

Domain of dependence for

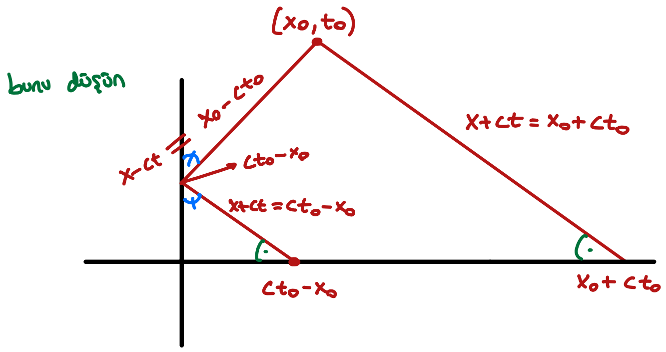
$$u_{tt} - c^2 u_{xx} = 0 \qquad x>0 \quad t>0$$
$$u(x,0) = f(x) \qquad x>0$$
$$u_t(x,0) = g(x) \qquad x>0$$
$$u(0,t) = 0$$

to find $u(x_0,t_0)$ we need f, g

if $x_0 - ct_0 > 0$ on $[x_0 - ct_0, \, x_0 + ct_0]$

if $x_0 - ct_0 < 0$ on $[ct_0 - x_0, \, x_0 + ct_0]$

Consider (x_0, t_0) s+ $x_0 - ct_0 < 0$



ex $\quad u_{tt} - c^2 u_{xx} = 0 \qquad t>0 \quad x>0$

$$u(x,0) = f(x) \qquad x>0$$
$$u_t(x,0) = g(x) \qquad x>0$$
$$u_x(0,t) = 0$$

Similarly to prev problem

$$u = F(x-ct) + G(x+ct)$$

$$G(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz \qquad \text{is well defined}$$

$$F(x-ct) = \frac{f(x-ct)}{2} + \frac{1}{2c}\int_0^{x-ct} g(z)\,dz \qquad \text{is NOT well defined if } x-ct<0$$

we need F for negative argument. Let us use boundary cond.

$$u_x(0,t) = 0 \Rightarrow F'(x-ct)\Big|_{x=0} + G'(x+ct)\Big|_{x=0} = 0$$

$$F'(-ct) = -G'(ct) \quad , \text{ say } -ct = z$$

$$F'(z) = -G'(-z)$$ Ask

$$\Downarrow$$

$$F(z) = G(-z) + A \qquad z<0$$

Now we can write u

$$u(x,t) = \begin{cases} \dfrac{f(x+ct)+f(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(z)\,dz & x-ct>0 \\[2ex] \dfrac{f(x+ct)+f(ct-x)}{2} + \dfrac{1}{2c}\displaystyle\int_0^{x+ct} g(z)\,dz + \dfrac{1}{2c}\displaystyle\int_0^{ct-x} g(z)\,dz + A & x-ct<0 \end{cases}$$

Since $u(x,t)$ is cont

$$u(x,t)\Big|_{x-ct\to 0^+} = u(x,t)\Big|_{x-ct\to 0^-} \Rightarrow A = 0$$

10 Nisan Perşembe

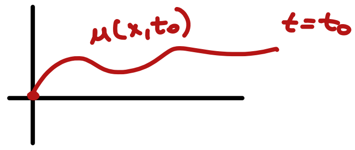
Recall we consider initial – boundary value problems. We considered the problem for a string with fixed end

I $\quad u_{tt} - c^2 u_{xx} = 0 \qquad t>0 \quad x>0$

$u(x,0) = f(x) \qquad x>0$

$u_t(x,0) = g(x) \qquad x>0$

$u(0,t) = 0$ (the left end of the string is fixed)



$u(x,t)$ is displacement of "pt x" at time "t"

$u(0,t) = 0 \ \forall t \Rightarrow$ the pt of the string with x-coordinate equal to zero has no displacement is fixed.

Ask types

we considered the problem for a string with free end

II $\quad u_{tt} - c^2 u_{xx} = 0 \qquad t>0 \quad x>0$

$u(x,0) = f(x) \qquad x>0$

$u_t(x,0) = g(x) \qquad x>0$

$u_x(0,t) = 0 \qquad$ free end

if $u_x(x_0,t) \neq 0$ then string "pulled along" x-axis, $\exists$ a tension

if $u_x(x_0,t) = 0$ then string has no tension at a pt with x-coord. zero (free)

Ask

we derived soln in particular for 2nd problem

free end $u_x(0,t)=0$

$$u(x,t) = \begin{cases} \dfrac{f(x-ct)+f(x+ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(z)\,dz & x-ct \geq 0 \\[2ex] \dfrac{f(ct-x)+f(x+ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{0}^{x+ct} g(z)\,dz + \dfrac{1}{2c}\displaystyle\int_{0}^{ct-x} g(z)\,dz & x-ct<0 \end{cases}$$

Remark if $f(x)$ and $g(x)$ are even, then soln of $u_{tt} - c^2 u_{xx} = 0 \quad t>0 \quad -\infty<x<\infty$

$u(x,0) = f(x) \qquad -\infty<x<\infty$

$u_t(x,0) = g(x) \qquad -\infty<x<\infty$

would also be even wrt x ($\forall t_0 \ \ u(x,t_0) = u(-x,t_0)$)

here, $u_x(0,t) = 0$ ($u_x(x,t)$ odd wrt x) Ask

So we can solve the problem II by extending $f(x)$ and $g(x)$ on $(-\infty, \infty)$ as <u>even fns</u>

then solve the resulting Cauchy problem on $(-\infty, \infty)$.

The soln will satisfy all cond of the problem II

3 types

1) no boundary / inf string D'Alemb

yes bdry Semi $\to$ fixed $u(0,t)=0$; free $u_x(0,t)=0$

ex Solve the problem

$$u_{tt} - 9u_{xx} = 0 \qquad x>0 \quad t>0$$

$$u(x,0) = \sin x \qquad x>0$$

$$u_t(x,0) = \cos 5x \qquad x>0$$

$$u_x(0,t) = 0 \qquad t>0$$

by using above formula

$$u(x,t) = \begin{cases} \dfrac{\sin(x-3t)+\sin(x+3t)}{2} + \dfrac{1}{6}\displaystyle\int_{x-3t}^{x+3t} \cos(5z)\,dz & x-3t \geq 0 \\[2ex] \dfrac{\sin(3t-x)+\sin(x+3t)}{2} + \dfrac{1}{6}\displaystyle\int_0^{3t-x} \cos(5z)\,dz + \dfrac{1}{6}\displaystyle\int_0^{x+3t} \cos(5z)\,dz & x-3t<0 \end{cases}$$

Non-Homogenous Wave Eqn

$$u_{tt} - c^2 u_{xx} = F(x,t) \qquad t>0 \qquad -\infty<x<\infty$$

$$u(x,0) = f(x) \qquad -\infty<x<\infty$$

$$u_t(x,0) = g(x) \qquad -\infty<x<\infty$$

we will consider two problems :

1. non-homogenous wave eqn with homogenous boundary cond.

$$v_{tt} - c^2 v_{xx} = F(x,t) \qquad t>0 \qquad x\in\mathbb{R}$$

$$v(x,0) = 0 \qquad x\in\mathbb{R}$$

$$v_t(x,0) = 0 \qquad x\in\mathbb{R}$$

2. Homogenous wave eqn with non-homogenous boundary cond.

$$w_{tt} - c^2 w_{xx} = 0 \qquad t>0 \qquad x\in\mathbb{R}$$

$$w(x,0) = f(x) \qquad x\in\mathbb{R}$$

$$w_t(x,0) = g(x) \qquad x\in\mathbb{R}$$

it's easy to see $u = v + w$ solves original problem

Recall (Green's Thm)

let Ω be a domain in $\mathbb{R}^2$ and $P, Q \in C^1(\Omega)$ then

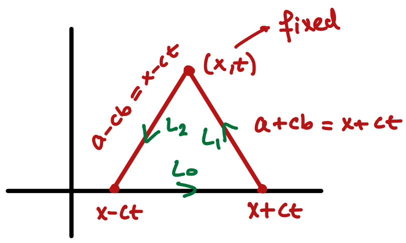
$$\iint_{\Omega} (P_x - Q_y)\, dx\,dy = \int_{\partial\Omega} P\,dy + Q\,dx \qquad (\partial\Omega \text{ oriented properly})$$

Thm the soln of the problem

$$u_{tt} - c^2 u_{xx} = F(x,t) \qquad t>0 \qquad -\infty < x < \infty$$

$$u(x,0) = 0 \qquad -\infty < x < \infty$$
$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

is given by $\quad u(x,t) = \dfrac{1}{2c} \displaystyle\iint_D F(a,b)\, da\, db \quad$ where D is characteristic triangle



Pf take ∂D oriented counter clock wise $\quad \partial D = L_0 \cup L_1 \cup L_2$

$$L_0 = \{(a,b) : b=0 \;,\; x-ct \le a \le x+ct\}$$

Ask

$$L_1 = \{(a,b) : a+cb = x+ct \quad 0 \le b \le t\}$$

$$L_2 = \{(a,b) : a-cb = x-ct \quad 0 \le b \le t\}$$

now we integrate $u_{tt} - c^2 u_{xx} = F(x,t)$ over D.

$$\iint_D \left(u_{bb}(a,b) - c^2 u_{aa}(a,b)\right) da\,db = \iint_D F(a,b)\, da\,db$$

By Green's Thm

$$\iint_D (u_{aa} - c^2 u_{bb}\; da\,db) = \int_{\partial D} -u_b\,da - c^2 u_a\,db$$

$$= \int_{L_0} -u_b\,da - c^2 u_a\,db + \int_{L_1} -u_b\,da - c^2 u_a\,db + \int_{L_2} -u_b\,da - c^2 u_a\,db$$

Consider each integral

1) $\displaystyle\int_{L_0} -u_b\,da - c^2 u_a\,db$

On L_0 $\quad b=0 \quad db=0 \quad$ and $\quad u_b = 0$

$$\int_{L_0} -u_b\,da - c^2 u_a\,db = 0$$

2) $\displaystyle\int_{L_1} -u_b\,da - c^2 u_a\,db = c\,u(x,t)$

On L_1 $\quad a+cb = x+ct \quad \rightarrow \quad da + c\,db = 0 \qquad da = -c\,db, \quad db = \frac{-1}{c}\,da$

$$= \int_{L_1} c u_b\,db + c u_a\,da = \int_{L_1} d(c u_b + c u_a)$$

$$= c\int_{L_1} du = c.u\Big|_{(x+ct,0)}^{(x,t)} = c\,u(x,t) - \underbrace{c\,u(x+ct,0)}_{0} = c\,u(x,t)$$

3) $\displaystyle\int_{L_2} -u_b\,da - c^2 u_a\,db$

On L_2 $\quad a-cb = x-ct \quad \Rightarrow \quad da - c\,db = 0 \qquad da = c\,db, \quad db = \frac{1}{c}\,da$

$$\int_{L_2} -u_b\,da - c^2 u_a\,db = \int_{L_2} -c u_b\,db - c u_a\,da$$

Ast

$$= -c\int_{L_2} u_b\,db + u_a\,da = -c\int_{L_2} du = -c u\Big|_{(x,t)}^{(x-ct,0)}$$

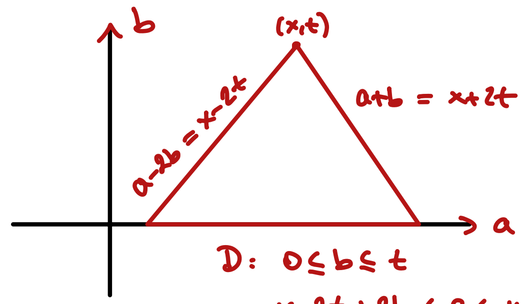
$$= -c\left(\overset{0}{u(x-ct,0)} - u(x,t)\right) = c\,u(x,t)$$

Combine all integrals, we get

$$0 + c\,u(x,t) + c\,u(x,t) = \iint_D F(a,b)\,da\,db \Rightarrow u = \frac{1}{2c}\iint_D F(a,b)\,da\,db \qquad \square$$

ex Solve the problem

$$u_{tt} - 4u_{xx} = xt \qquad c=2$$
$$u(x,0) = 0$$
$$u_t(x,0) = x$$



D: $0 \le b \le t$

$x-2t+2b \le a \le x+2t-2b$

it follows
$$u(x,t) = \frac{0+0}{2} + \frac{1}{2.2}\int_{x-2t}^{x+2t} z\,dz + \frac{1}{2.2}\iint_D ab\,da\,db$$

$$u(x,t) = \frac{1}{4}\,\frac{z^2}{2}\Big|_{x-2t}^{x+2t} + \frac{1}{4}\int_0^t\left(\int_{x-2t+2b}^{x+2t-2b} a\,b\,da\right)db$$

$$= \frac{(x+2t)^2-(x-2t)^2}{8} + \frac{1}{4}\int_0^t b\,\frac{a^2}{2}\Big|_{x-2t+2b}^{x+2t-2b}\,db$$

$$= \frac{x^2+4xt+4t^2-x^2+4xt-4t^2}{8} + \frac{1}{8}\int_0^t b\left((x+2t-2b)^2-(x-2t+2b)^2\right)db$$

$$= xt + \frac{1}{8}\int_0^t b\,(8xt-8xb)\,db$$

$$= xt + xt\,\frac{b^2}{2}\Big|_0^t - x\,\frac{b^3}{3}\Big|_0^t = xt + \frac{xt^3}{2} - \frac{xt^3}{3} = xt + \frac{xt^3}{6} \qquad \square$$

Energy Integral

Consider

$$u_{tt} - c^2u_{xx} = 0 \qquad t>0 \qquad -\infty<x<\infty$$
$$u(x,0) = f(x) \qquad -\infty<x<\infty$$
$$u_t(x,0) = g(x) \qquad -\infty<x<\infty$$

Define $$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_t^2 + c^2u_x^2\right)dx$$

Lemma let $f, g \in C^2(\mathbb{R})$ suppose

f, g are zero outside $[-N, N]$

Ast

$(f(x) = 0 \;,\; g(x) = 0$ if $|x| > N)$

Then $E(t)$ is constant

$$\frac{d}{dt} E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (2 u_t u_{tt} + c^2\, 2 u_x u_{xt})\, dx$$

$$= c^2 \int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{xt})\, dx$$

$$= c^2 \int_{-\infty}^{\infty} (u_t \cdot u_x)_x\, dx$$

$$= c^2 u_t \cdot u_x \Big|_{-\infty}^{\infty}$$

Note since f, g are zero outside $[-N, N]$ for any $t > 0$ (fixed)

if $|x|$ is large enough

the domain of dependence for (x, t)

$[x-ct, x+ct] \cap [-N, N] = \phi$

So $u(x,t) = 0 \Rightarrow u_x(x,t) = 0$

Hence $\frac{d}{dt} E(t) = 0$

15 Nisan Sali

Recall we consider

$$u_{tt} - c^2 u_{xx} = 0 \qquad -\infty < x < \infty \qquad t > 0$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = g(x) \qquad -\infty < x < \infty$$

we define Energy Integral

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\, dx$$

we proved when $f(x)$, $g(x)$ are zero

outside $[-N, N]$ $(\text{supp } f, \text{ supp } g \in [-N, N])$ then $E(t)$ is constant

we need a lemma from Calculus

lemma let $h(x)$ be cont non-negative fn on $\mathbb{R}$ if

$$\int_a^b h(x)dx \quad \text{then} \quad h(x) = 0 \quad \forall x \in [a,b]$$

Thm (the Cauchy Problem) $u(\cdot\cdot, 0)$ ↳ time $= 0$

$$u_{tt} - c^2 u_{xx} = F(x,t) \qquad -\infty < x < \infty, \ t > 0$$
$$u(x,0) = f(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = g(x) \qquad -\infty < x < \infty$$

has at MOST 1 SOLUTION !

pf assume that u_1, u_2 are two solns. Then $v = u_1 - u_2$ is a soln too and satisfy

$$\left.\begin{aligned} &v_{tt} - c^2 v_{xx} = 0 \\ &v(x,0) = 0 \\ &v_t(x,0) = 0 \end{aligned}\right\} \longrightarrow \text{ so } E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (v_t^2 + c^2 v_x^2)\, dx \quad \text{is constant}$$

how we know zero

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} (\underbrace{v_t^2(x,0)}_{0} + c^2 \underbrace{v_x^2(x,0)}_{0})\, dx = 0 \xrightarrow{\text{we have}} \forall t \quad E(t) = 0$$

$$\frac{1}{2}\int_{-\infty}^{\infty} (v_t^2 + c^2 v_x^2)\, dx = 0.$$ By prev lemma

so $\forall t$ $v_t^2 + c^2 v_x^2 = 0$ on $(-\infty, \infty)$

note: $v(x,0) = 0 \rightarrow \frac{\partial}{\partial x} v(x,0) = 0$

it follows $v_t = 0$, $v_x = 0 \Rightarrow v(x,t) = c$, since $v(x,0) = 0 \Rightarrow v(x,t) = 0$, $\forall t > 0$ $-\infty < x < \infty$

$v = 0 \Rightarrow u_1 - u_2 = 0 \Rightarrow u_1 = u_2$ here the soln is unique ex Show that the initial boundary problem has at most 1 soln

$$u_{tt} + k^2 u_t - c^2 u_{xx} \qquad t>0 \quad 0<x<\ell$$
$$c, k \in \mathbb{R}$$

$u(x,0) = f(x)$
$u_t(x,0) = g(x)$
$u(0,t) = 0 \quad , \quad u(\ell,t) = 0$

we consider Energy integral

$$E(t) = \frac{1}{2}\int_0^\ell (u_t^2 + c^2 u_x^2)\, dx$$

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^\ell (2u_t \underbrace{u_{tt}}_{-k^2 u_t + c^2 u_{xx}} + 2c^2 u_x u_{xt})\, dx$$

$$= \int_0^\ell (-k^2 u_t^2 + c^2 u_{xx} + c^2 u_x u_{xt})\, dx = -k^2 \int_0^\ell u_t^2\, dx + c^2 \int_0^\ell (u_t u_x)_x\, dx$$

$$= -k^2 \int_0^\ell u_t^2\, dx + c^2 u_t u_x \Big|_0^\ell = -k^2 \int_0^\ell u_t^2\, dx$$

$u(0,t) = 0 \quad , \quad u(\ell,t) = 0 \quad \Rightarrow \quad u_t(0,t) = 0 \qquad u_t(\ell,t) = 0$

we have $\quad \frac{d}{dt}E(t) = -k^2 \int_0^\ell u_t^2\, dx \leq 0$

So, $E(t)$ non-increasing $\qquad E(t) \leq E(0) \qquad E>0$

assume $u_1 - u_2$ are two soln . $\quad v = u_1 - u_2$ we have

$v_{tt} + k^2 v_t - c^2 v_{xx} = 0$
$v(x,0) = 0$
$v(x,0) = 0$
$v(0,t) = 0 \quad v(\ell,t) = 0$

hence $E(t) = \frac{1}{2}\int_0^\ell (v_t^2 + c^2 v_x^2)\, dx \qquad$ then $\quad E(t) \leq E(0) \quad \forall t>0$

$$E(0) = \frac{1}{2}\int_0^\ell (\underbrace{v_t^2(x,0)}_{0} + c^2 \underbrace{v_x^2(x,0)}_{0}) = 0$$

$v(x,0) = 0 \Rightarrow v_x(x,0) = 0 \qquad \forall t \quad 0 \leq E(t) \leq E(0) \quad \rightarrow \quad E(t) = 0$

In the same way as did for wave eqn

we have $v = 0 \rightarrow u_1 = u_2$ the soln is unique

HEAT EQUATION (parabolic eqn)

(*) $u_t^2 - k^2 u_{xx} = F(x,t)$ $\quad 0<x<\ell \quad t>0 \quad k\in\mathbb{R}$

$u(x,0) = f(x)$

$u(0,t) = \alpha(t) \qquad u(\ell,t) = \beta(t)$

the uniqueness of soln

energy integral define $E(t) = \frac{1}{2}\int_0^\ell (u^2\,dx)$

we assume u satisfies

$$u_t - k^2 u_{xx} = 0$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0 \qquad u(\ell,t) = 0$$

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^\ell 2u\underbrace{u_t}_{k^2u_{xx}}\,dx = \int_0^\ell u\,u_{xx}\,dx = k^2\overset{0}{\cancel{u\,u_x}}\Big|_0^\ell - k^2\int_0^\ell u_x^2\,dx = -k^2\int_0^\ell u_x^2\,dx < 0$$

IBP: $\left.\begin{array}{ll} p = u & q = u_x \\ p_x = u_x & q_x = u_{xx} \end{array}\right\}$

$\Rightarrow E(t)$ not increasing

$E(t) < E(0)$

let u_1, u_2 be two solns $\quad v = u_1 - u_2$ satisfies

$$v_t - k^2 v_{xx} = 0$$
$$v(x,0) = 0$$
$$v(0,t) = 0 \qquad v(\ell,t) = 0$$

for $E(t) = \frac{1}{2}\int_0^\ell v^2 dx$ we have $E(0) = \frac{1}{2}\int_0^\ell \underbrace{v^2(x,0)}_{0}\,dx = 0$ so $0 \le E(t) \le E(0) = 0$

$E(t) = 0$

$E(t) = \int_0^\ell v^2 dx \to v^2 = 0 \to v = 0 \qquad$ so $u_1 = u_2$ we have unique soln

Similarly the original problem (*) has unique soln

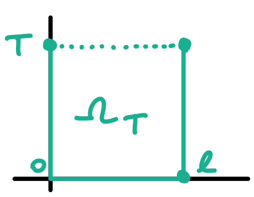
Maximum Principle

Define $\Omega_T = \{(x,t) : 0<x<\ell,\ 0<t\le T\}$ open domain in $\mathbb{R}^2$. For Ω_T we define

parabolic boundary $\quad B_T = \{(x,t) : t=0,\ 0\le x\le L$ or $x=0 \quad 0\le t\le T$

or $x=\ell \quad 0\le t\le T$

note $B_T \subsetneqq \partial\Omega_T$



Thm (Weak Maximum Principle)

let a fn $u(x,t)$, $u \in C^2(\Omega_T)$ satisfies

$$u_t - k^2 u_{xx} \leq 0 \quad \text{in } \Omega_T \quad \text{Then}$$

$$\max_{\bar{\Omega}_T} u = \max_{B_T} u$$

pf first we suppose $u_t - k^2 u_{xx} < 0$ in Ω_T

we want to show that $\max\limits_{\bar{\Omega}_T} u = \max\limits_{B_T} u$

assume NOT true $\max\limits_{\bar{\Omega}_T} u \neq \max\limits_{B_T} u$

$\bar{\Omega}_T$ closed bdd set and $u(x,t)$ is cont.

u has max value at (x_0, t_0) Also $(x_0, t_0) \notin B_T$

we can have $(x_0, t_0) \in \Omega_T$ (interior pt) then $u_t(x_0,t_0) = 0$ and $u_{xx}(x_0,t_0) \leq 0$

hence $u_t(x_0,t_0) - k^2 u_{xx}(x_0,t_0) \geq 0$ contradiction, not possible

we have $u_t - k^2 u_{xx} < 0$ in Ω_T

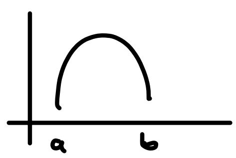
we can have $(x_0,t_0) \in \{(x,t) : t = T \quad 0 < x < l\}$

then $u_t(x_0,t_0) \geq 0$ same for $u_{xx}(x_0,t_0) \leq 0 \Rightarrow u_t(x_0,t_0) - k^2 u_{xx}(x_0,t_0) \geq 0$

not possible

$f(x)$ on $[a,b]$

f has max value at $x_0 \in (a,b)$



f is concave up about $x_0 \Rightarrow f''(x_0) \leq 0$

our assumption that $\max\limits_{\bar{\Omega}_T} u \neq \max\limits_{B_T} u$ leads to contradiction so its wrong

Hence $\max\limits_{\bar{\Omega}_T} u = \max\limits_{B_T} u$

now consider general case $u_t - k^2 u_{xx} \le 0$ in Ω_T

consider $v(x,t) = u(x,t) - \varepsilon t$ $\quad \varepsilon > 0$

$$v_t - k^2 v_{xx} = u_t - \varepsilon - k^2 u_{xx} = u_t - k^2 u_{xx} - \varepsilon < 0$$

here $\max_{\bar{\Omega}_T} v = \max_{B_T} v$

note $u \geq v$ in Ω_T $\rightarrow$ $\max_{B_T} u > \max_{B_T} v = \max_{\bar{\Omega}_T} v = u - \varepsilon t$ $\quad \forall (x,t) \in \Omega_T$

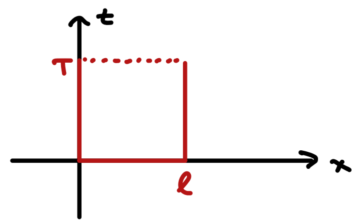
$\max_{B_T} u \geq u - \varepsilon t$ $\quad \forall (x,t) \in \Omega_T$ $\quad$ true $\forall \varepsilon > 0$

take $\varepsilon \to 0^+$ we get $\max_{B_T} u \geq u$ $\quad \forall (x,t) \in \bar{\Omega}_T$

$\max_{B_T} u \geq \max_{\bar{\Omega}_T} u$ $\quad$ and $\quad B_T < \bar{\Omega}_T$

$\max_{\bar{\Omega}_T} u \geq \max_{B_T} u$ $\quad$ so, $\quad \max_{\bar{\Omega}_T} u = \max_{B_T} u$

17 Nisan Pers



Recall we introduced $\Omega_T = \{ (x,t) : 0 < x < \ell \,,\; 0 < t < T \}$

$B_T \subsetneqq \partial \Omega_T$

and $B_T = \{ (x,t) : t = 0 \,,\; 0 \le x \le t \;\text{ or }\; x = 0 \,,\; 0 \le t \le T \;\text{ or }\; x = \ell \,,\; 0 \le t \le T \}$

we proved

Thm (weak max principle for heat eqn)

suppose that a fn $u(x,t)$ satisfies inequality

$$u_t - k^2 u_{xx} \le 0 \text{ in } \Omega_T \,,\; u \in C^2(\bar{\Omega}_T) \,.\; \text{Then } \max_{\bar{\Omega}_T} u = \max_{B_T} u$$

Remark the maximum principle can formulated as follow. Suppose $u \in C^2(\bar{\Omega}_T)$ satisfies $u_t - k^2 u_{xx} \leq 0$, $k \neq 0$. If $u < A$ $\forall (x,t) \in B_T$ then $u < A$ $\forall (x,t) \in \bar{\Omega}_T$

ex let u be smooth fn satisfying

$$u_t - u_{xx} = 0 \qquad 0<x<\pi \qquad 0<t<T \qquad T>0$$

$$u(x,0) = \sin^2 x$$

$$u(0,t) = 0 \qquad u(\pi,t) = 0$$

Show that $u \leq e^{-t} \sin x$ for $0 \leq x \leq \pi$, $0 \leq t \leq T$

consider $V = u - e^{-t} \sin x$

$$V_t - V_{xx} = (u_t + e^{-t}\sin x) - (u_{xx} + e^{-t}\sin x) = 0 \leq 0 \qquad \text{in } \Omega_T$$

Note $\Omega_T = \{(x,t) : 0<x<\pi, 0<t<T\}$

we check v on B_T

$x=0$ $\quad 0<t<T \qquad v(0,t) = u(0,t) - e^{-t}\sin 0 = 0$

$x=\pi$ $\quad 0 \leq t \leq T \qquad v(\pi,t) = u(\pi,t) - e^{-t}\sin\pi = 0$

Ask about inequalities $\leq$ $<$

$0 \leq x \leq \pi$ $\quad t=0 \qquad v(x,0) = u(x,0) - \sin x = \sin^2 x - \sin x \leq 0$ on $[0,\pi]$

hence $V \leq 0$ in B_T by the maximum principle $V \leq 0$ in $\bar{\Omega}_T$

so, $u \leq e^{-t} \sin x$

Corollary $(\overset{\min}{u_t - u_{xx}} \leq 0, \ \overset{\max}{-u_t + u_{xx}} \leq 0)$

let $u \in C^2(\bar{\Omega}_T)$ satisfies

$$u_t - k^2 u_{xx} = 0 \quad \text{in } \Omega_T \qquad \text{then}$$

1) $\max\limits_{\bar{\Omega}_T} u = \max\limits_{B_T} u$, $\min\limits_{\bar{\Omega}_T} u = \min\limits_{B_T} u$

2) $\max\limits_{\bar{\Omega}_T} |u| = \max\limits_{B_T} |u|$

$u_t - k^2 u_{xx} = 0$ in Ω_T

1) $\max_{\bar\Omega_T} u = \max_{B_T} u$, $\min_{\bar\Omega_T} u = \min_{B_T} u$ 2) $\max_{\bar\Omega_T} |u| = \max_{B_T} |u|$

Pf 1) recall if u has max and min values on a set A then

$$-\min_A u = \max_A (-u)$$

hence $-\min_{B_T} u = \max_{B_T} (-u) = \max_{\bar\Omega_T} (-u) = -\min_{\bar\Omega_T} (u)$ by Maximum principle
$\left((-u_t) - (-u_{xx}) \leq 0 \right)$

so $\min_{B_T} u = \min_{\bar\Omega_T} u$

Pf 2) directly follows from 1)

$$\max_A |u| = \max \{ |\max_A u| , |\min_A u| \}$$

Thm uniqueness the initial boundary value problem

$$u_t - k^2 u_{xx} = F(x,t) \qquad 0 < x < l \qquad t > 0$$
$$u(x,0) = f(x) \qquad 0 \leq x \leq l$$
$$u(0,t) = \alpha(t) \qquad u(l,t) = \beta(t) \qquad t \geq 0$$

where F, f, α, β are smooth , has at most one soln

Pf let u_1, u_2 be two solns then $v = u_1 - u_2$ satisfies

$$v_t - k^2 v_{xx} = 0 \qquad \text{in } \Omega_T$$
$$v(x,0) = 0 \qquad 0 \leq x \leq l$$

$v(0,t) = 0$, $v(l,t) = 0$ $\quad 0 \leq t \leq T$ for any $T > 0$, by the Maximum principle

$\max_{\bar\Omega_T} |v| = \max_{B_T} |v|$ and $v|_{B_T} = 0 \Rightarrow \max_{\bar\Omega_T} |v| = 0$

so $\forall T > 0$ $v = 0$ in $\bar\Omega_T$. Hence, $v = 0$ $\forall\, 0 \leq x \leq l$, $t > 0$ so $u_1 - u_2 = 0$

$\forall\, 0 \leq x \leq l$ $t > 0$. The soln is unique if exists

Thm Stability let $u(x,t)$ be a soln of the problem

$$u_t - k^2 u_{xx} = F(x,t) \qquad 0<x<\ell \ , \ t>0$$
$$u(x,0) = f(x) \qquad 0\le x\le \ell$$
$$u(0,t) = \alpha(t) \ , \ u(\ell,t) = \beta(t) \qquad t>0$$

F, f, α, β smooth in their domain

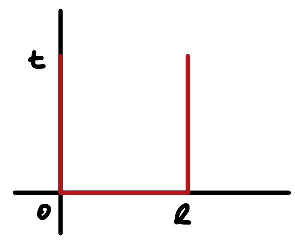
then for any $T>0$ if $\max\limits_{B_T} u = M$ and $|F(x,t)| < N$ in $\overline{\Omega}_T$, then

$$|u| \le M + N.T \qquad \text{in } \overline{\Omega}_T$$

Pf consider $v = u - (M + Nt)$

$$v_t - k^2 v_{xx} = (u_t - k^2 u_{xx}) - N \le 0 \quad \text{in } \Omega_T \quad (|F|\le N \text{ in } \Omega_T)$$

$$v = u - (M + At) < u - (M + AT) < 0 \Rightarrow u < M + At \quad \text{on } B_T$$



By maximum principle on B_T

$v = u - (M + Nt) < 0 \quad \forall\ 0\le t\le T$ then by Max principle on $\overline{\Omega}_T$

$\Rightarrow v = u - (M+Nt) \le 0$ (since $0 < (M+Nt) < (M+NT)$ in $\overline{\Omega}_T$) $\to u - (M+NT) \le 0$

in the same way applying the Maximum principle to $-u - (M + Nt)$ we get

$$-u \le M + NT \quad \text{in } \overline{\Omega}_T$$

hence, $|u| < M + NT$ ▯

corollary (Stability of soln for heat eqn PDE)

let u be smooth fn that satisfies

$$u_t - k^2 u_{xx} = F(x,t) \qquad 0<x<\ell \qquad t>0$$
$$u(x,0) = f(t) \qquad 0\le x\le \ell$$
$$u(0,t) = \alpha(t) \ , \ u(\ell,t) = \beta(t) \qquad t\ge 0$$

if $|f(x)| < \varepsilon$, $0\le x\le \ell$ $\quad |\alpha(t)| < \varepsilon$, $|\beta(x)| < \varepsilon$ $\quad t\ge 0$ and $|F(x,t)| < \varepsilon$

then $\forall\ T>0 \qquad |u| \le \varepsilon(1+T) \quad \forall\ 0\le x\le \ell \quad 0\le t\le T$

MT-2
Wave Eqn
* Max principle
* idea on Seperation of Var
No Sep. of Var Heat Eqn

SEPARATION OF VARIABLES

consider $$u_t - k^2 u_{xx} = 0 \qquad 0<x<\ell \qquad t>0$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0 \qquad u(\ell,t) = 0 \qquad t \geq 0$$

we assume $u = X(x) \cdot T(t)$ then subst u into the eqn

$$u_t = X \cdot T' - k^2 X'' T = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{k^2} \frac{T'}{T} = -\lambda$$ right part fn of t only, left part fn of x only

so both sides are constant. We have two eqns

1) $X'' + \lambda X = 0$

2) $T'' + k^2 \lambda T = 0$

if X and T satisfy the above ODE then $u = X(x)\,T(t)$ satisfies the original PDE

we look for soln satisfying $u(0,t) = 0$ $\quad u(\ell,t) = 0 \quad t>0$

$$u(0,t) = X(0) \cdot \underset{\neq 0}{T(t)} = 0 \quad \Rightarrow \quad X(0) = 0$$

$$u(\ell,t) = X(\ell) \cdot \underset{\neq 0}{T(t)} = 0 \quad \Rightarrow \quad X(\ell) = 0$$

hence for X we have boundary value problem

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X(\ell) = 0$$

remark: the above boundary value ODE can be considered as eigenvalue problem for $A = \frac{d^2}{dx^2}$ on space $E = \{ g : g \in C^2[0,\ell] \quad g(0) = g(\ell) = 0 \}$

we need all λ st $\exists\, g \neq 0$ satisfying the cond

$$Ag = -\lambda g$$

lets solve $X'' + \lambda X = 0 \qquad X(0) = 0 \quad X(\ell) = 0$

1) $\lambda = 0$ $\quad X'' = 0 \quad X = c_1 x + c_2$

$$\left.\begin{aligned} X(0) = 0 &\Rightarrow c_2 = 0 \\ X(\ell) = 0 &\Rightarrow c_1 = 0 \end{aligned}\right\} X = 0 \quad \text{only trivial soln}$$

2) $\lambda > 0$ $\quad \lambda = \mu^2$

$X'' + \mu^2 X$

$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r = \mp \mu i$

$X = c_1 \cos \mu x + c_2 \sin \mu x \qquad X(0) = 0 \Rightarrow c_1 = 0$

$X(\ell) = 0 \Rightarrow c_2 = \sin \mu \ell = 0$

we need $c_2 \neq 0$ for $X \neq 0$

$\sin \mu \ell = 0 \qquad \mu \ell = \pi k \qquad k = 1, 2, 3$

we have eigenvalues $\lambda_k = \left(\frac{\pi k}{\ell}\right)^2$, $k = 1, 2, 3 \ldots$ with correspond

eigenfunction $X_k = \sin\left(\frac{\pi k}{\ell} x\right)$

the eigenfunction is determined up to a constant

$\lambda < 0$ $\quad \lambda = -\mu^2 \qquad \mu >$

$X'' - \mu^2 X = 0$

$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r = \mp \mu \quad \Rightarrow \quad X = c_1 e^{\mu t} + c_2 e^{-\mu t}$

$$\left.\begin{aligned} X(0) = 0 &\rightarrow c_1 + c_2 \\ X(\ell) = 0 &\rightarrow c_1 e^{\mu \ell} + c_2 e^{-\mu \ell} = 0 \end{aligned}\right\} \begin{pmatrix} 1 & 1 \\ e^{\mu \ell} & e^{-\mu \ell} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} 1 & 1 \\ e^{\mu \ell} & e^{-\mu \ell} \end{pmatrix} \neq 0 \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \qquad c_1 = c_2 = 0 \quad \text{only trivial soln}$$

we found that $X'' + \lambda X = 0 \quad X(0) = 0 \quad X(\ell) = 0$ has eigenvalues $\lambda_k = \left(\frac{\pi k}{\ell}\right)^2$

with eigenfunctions $X_k = \sin\left(\frac{\pi k}{\ell}\right) \qquad k = 1, 2 \ldots$

22 Nisan Salı

Recall, Separation of Variables

ex replace given PDE by ODE if possible

a) $u_{xx} + u_{tt} + xu = 0$

b) $u_{xx} + (x+y) u_{yy} = 0$

c) $a^2 (u_{xx} + u_{yy}) = u_t$

Soln a) $u(x,t) = X(x) T(t)$

then $X'' \cdot T + T'' X + x X T = 0$

we dont want x and T to be in same $(x+T)$ (xT) NOT GOOD

$T(X'' + xX) = -X T''$

$$\frac{X'' + xX}{X} = -\frac{T''}{T} = -\lambda$$

$\lambda \in \mathbb{R}$

LHS x only
RHS y only

$$\left.\begin{aligned} X'' + xX &= -\lambda X \\ T'' &= \lambda T \end{aligned}\right\} \text{Two ODE}$$

if X and T satisfy

$$X'' + (x+\lambda) X = 0$$

and

$$T'' - \lambda T = 0$$

then $u = XT$ satisfies $u_{xx} + u_{tt} + xu = 0$

ex replace given PDE by ODE if possible

b) $u_{xx} + (x+y)\, u_{yy} = 0$

say $u = X(x)\, Y(y)$

$$X''Y + (x+y) X Y'' = 0$$

we can have

$$\frac{X''}{(x+y)X} = -\frac{Y''}{Y}$$ variables are NOT separated

or $$\frac{X''}{X} = -\frac{Y''}{Y}(x+y)$$ variables are NOT separated

Sepr of Var not usable

ex replace given PDE by ODE if possible

c) $a^2 (u_{xx} + u_{yy}) = u_t$

say $u = X(x)\, Y(y)\, T(t)$

$$a^2 (X''YT + XY''T) = XYT'$$

$$a^2 (X''Y + XY'')T = XYT'$$

$$\frac{X''Y + XY''}{XY} = \frac{T'}{a^2 T} = -\lambda$$ LHS fn of (x,y) only, RHS fn of t only

Hence, constant.

we have

1 $\frac{T'}{a^2 T} = -\lambda$

separate this too 2 $\frac{X''Y + XY''}{XY} = -\lambda \Rightarrow \frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$

$\Rightarrow \frac{X''}{X} = -\mu$

$\frac{Y''}{Y} = \mu - \lambda$

if X, Y, T satisfy

1) $T' + \lambda a^2 T = 0$

2) $X'' + \mu X = 0$

3) $Y'' + (\lambda - \mu) Y = 0$

then $u = XYT$ satisfy $a^2 (u_{xx} + u_{yy}) = u_t$