

Math 358 PDE "for 1st order questions we'll use Sneddon's Elements of PDE"

lecture 1 18 Subat almost every 2 weeks open book quiz

PDE: an unknown fn that contains partial derivatives of an unknown fn is called PDE

the unknown fn is a fn of several variables

Defn the order of PDE is the order of highest derivative involved.

ex $u_x + u u_y = 0$ 1st order
 $u_{tt} + u_{xx} = 0$ 2nd order

linear PDE's

consider general n-th order PDE

$$F(x, y, u, u_x, u_y, \dots) = 0, \text{ to see linearity define an operator}$$

$$L(u) = F(x, y, u, \dots)$$

ex $u_x + u u_y = 0$ then $L(u) = u_x + u u_y$

$$u_t = u_{xxx} + u u_x \Leftrightarrow u_t - u_{xxx} - u u_x = 0 \Rightarrow L(u) = u_t - u_{xxx} - u u_x$$

if the operator L is linear that is $L(u+v) = L(u) + L(v)$ and $L(cu) = c \cdot L(u) \quad \forall c \in \mathbb{R}$
then PDE is linear

ex check if linear

a) $u_{tt} + u_{xx} = 0$ (linear) b) $u_t - u_{xxx} - u u_x = 0$ (non linear)

Soln a) $L(u) = u_{tt} + u_{xx}$

$$L(u+v) = (u+v)_{tt} + (u+v)_{xx} = u_{tt} + v_{tt} + u_{xx} + v_{xx} = L(u) + L(v)$$

$$L(cu) = (cu)_{tt} + (cu)_{xx} = c u_{tt} + c u_{xx} = c \cdot L(u)$$

Soln b) $L(u) = u_t - u_{xxx} - u u_x$

$$L(u+v) = (u+v)_t - (u+v)_{xxx} - (u+v)(u+v)_x$$

$$= u_t + v_t - u_{xxx} - v_{xxx} - u u_x - v u_x - u v_x - v v_x = L(u) + L(v) - v u_x - u v_x$$

remark let $L(u)$ be linear operator then

$L(u) = 0$: homogeneous linear eqn

$L(u) = f(x, y)$: non-homogeneous linear eqn

Superposition Principle

let $L(u) = 0$ be a linear eqn then for any u, v solns $L(u) = 0, L(v) = 0$

we have that $(c_1 u + c_2 v)$ $c_1, c_2 \in \mathbb{R}$ is also a soln

let $L(u) = f$ be non-homog eqn and u, v be two solns then $(u-v)$ is a soln

if the associated linear eqn $L(u) = 0$

Solns of PDE

consider n -th order eqn $F(x, u, u_x, \dots) = 0$

$u(x)$, $x = (x_1, \dots, x_n)$ in the domain $\Omega \subset \mathbb{R}^n$, we say u solves the eqn (classical soln)

if $u \in C^n(\Omega)$, u has continuous partial derivative up to order n , and the eqn

$F(x, u, u_x, u_{xx}, \dots)$ is satisfied identically in Ω

ex $u_t = u_{xx}$ $t > 0, x \in \mathbb{R}$ forward heat eqn

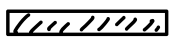
$\Omega = \{(x, t) : t \in (0, \infty), x \in (-\infty, \infty)\}$ "Well posed problem"

$u_t = u_{xx}$ $t < 0, x \in \mathbb{R}$ backward heat eqn

$\Omega = \{(x, t) : t \in (-\infty, 0), x \in (-\infty, \infty)\}$ "ill posed problem"



nosil yoyilacagını anlama kolay "well posed"



yayıldıkten sonra basta neresi yitmiş anlamak çok zor "ill posed"

ex for eqn $u_t = 4u_{xx} \quad t > 0 \quad x \in \mathbb{R}$, find a soln of the form $u = e^{(mt)} \sin(nx)$

note $u = e^{mt} \sin(nx) \in C^\infty(\Omega)$ where $\Omega = \{(x,t) : t > 0, x \in \mathbb{R}\}$

substit u into the eqn

$$\underbrace{m e^{mt} \sin(nx)}_{u_t} = 4 \underbrace{\left(-e^{mt} n^2 \sin(nx) \right)}_{u_{xx}} \Rightarrow m = -4n^2$$

for any $u = e^{-4n^2 t} \sin(nx)$ solves the eqn

what we can say about general soln of PDE?

ex $z_x = y + x \quad (x,y) \in \mathbb{R}^2$ integrate wrt x

$$z = \frac{1}{2}x^2 + \frac{y^2}{2} + C(y) \quad , \quad z \text{ contains arbitrary fn of } y$$

ex let $f \in C^\infty$ be an arbitrary fn, find a PDE of smallest possible order that has a soln

$$u = e^x f(x-2y) \quad (1)$$

↳ 1 arbitrary fn ~ 1st order?

$$u_x = e^x f(x-2y) + e^x f'(x-2y) \quad \text{note } \frac{d}{dx}(x-2y) = 1 \quad (2)$$

$$u_y = e^x f'(x-2y) \cdot (-2) \quad (3)$$

now we can express f and f' in terms of u , u_x , u_y and subst. into 3rd eqn

$$f(x-2y) = u e^{-x}$$

$$f' = u_x e^{-x} - f(x-2y) = u_x e^{-x} - u e^{-x}$$

$$u_y = e^x (u_x e^{-x} - u e^{-x}) \quad (-2)$$

$$u_y = -2u_x + 2u$$

$$u_y + 2u_x = 2u \quad : \quad \text{first order eqn}$$

ex $u_{xy} = 0 \quad (x,y) \in \mathbb{R}^2$

$$u_y = f(y)$$

$$u = \int f(y) dy + g(x)$$

$$u = h(y) + g(x) \quad \text{when } h, g \in C^2(\mathbb{R}) \text{ arbitrary}$$

remark we can expect a general soln of n th order eqn to contain n many arbitrary fns

what about unique soln? to have unique soln we need additional conditions

so called boundary or/and IVP

ex $u_{tt} = u_{xx}$ (wave eqn) $t > 0 \quad x \in \mathbb{R}$

$$\left. \begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \right\} \text{ IVP and conditions}$$

usually we want well posed problems and

- 1) existence
- 2) uniqueness
- 3) stability (small changes in initial/boundary cond. create small changes in the soln)

ex (non existence)

$$u \in C^2(\mathbb{R}) \quad ; \quad \begin{aligned} u_x &= 1 \\ u_y &= 1 \end{aligned} \quad \text{has no soln}$$

recall if $u \in C^2(\mathbb{R})$ then $u_{xy} = u_{yx}$ but

$$\left. \begin{aligned} u_{xy} &= (u_x)_y = 1 \\ u_{yx} &= (u_y)_x = 0 \end{aligned} \right\} 1 \neq 0 \Rightarrow u_{xy} \neq u_{yx}$$

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first order diff eqns (linear/quasilinear in $z(x,y)$, $(x,y) \in \Omega \subset \mathbb{R}^2$)

linear $A(x,y) z_x + B(x,y) z_y = C(x,y) \cdot z$

quasi-linear $A(x,y,z) z_x + B(x,y,z) z_y = C(x,y,z)$ linear in $z_x z_y$

we solve both in same way

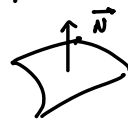
$$u(x,y,z) = C$$

$$z_x = -\frac{\mu_x}{\mu_z}, \quad z_y = -\frac{\mu_y}{\mu_z} \quad \left\{ \begin{array}{l} \mu_x + \mu_z \cdot z_x = 0 \end{array} \right.$$

method of characteristics

consider a surface $z = z(x,y)$, $(x,y) \in \Omega$ in $\mathbb{R}^3$ at a pt P on the surface

We have a normal vector $\vec{N} = (z_x, z_y, -1)$



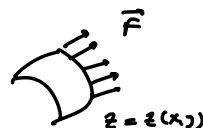
if we define a vector field

$$\vec{F} = A\vec{i} + B\vec{j} + C\vec{k}$$

the eqn $A z_x + B z_y = C$ OR $A z_x + B z_y - C = 0$ is exactly $\vec{F} \cdot \vec{N} = 0$

it means that the surface $z = z(x,y)$ is tangent to $\vec{F}$

to solve the eqn, we need to construct surfaces tangent to $\vec{F}$



we can construct integral curves for $\vec{F}$ curves tangent to $\vec{F}$ and then we

construct integral surface from these curves

take a curve $\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$\vec{r}(t)$ is tangent to curve $\vec{F}$ if $\frac{d\vec{r}}{dt} \parallel \vec{F} \Rightarrow \frac{d\vec{r}}{dt} = \alpha(t) \vec{F}$ at all pts of $\vec{r}$

$$\text{so, } \left. \begin{array}{l} x' = \alpha A \\ y' = \alpha B \\ z' = \alpha C \end{array} \right\} \frac{x'}{A} = \frac{y'}{B} = \frac{z'}{C} = \alpha \quad \text{OR reparametrize and get } \left. \begin{array}{l} x' = A \\ y' = B \\ z' = C \end{array} \right\} \text{system of ODE}$$

it is convenient to write the system as follows A, B, C are fns

$$\left(\frac{dx}{dt} = A, \quad \frac{dy}{dt} = B, \quad \frac{dz}{dt} = C \right) \quad \text{then} \quad \frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \quad (= dt)$$

one can get $\frac{dy}{dx} = \frac{B}{A}$ etc

recall that a fn $\varphi(x, y, z)$ called first integral of a system if φ is constant

on all solutions : $\varphi(x, y, z) = k, \quad k \in \mathbb{R}$

if we can find two functionally independent first integrals

$$\begin{cases} \varphi_1(x, y, z) = k_1 \\ \varphi_2(x, y, z) = k_2 \end{cases} \quad \text{then we have this integral curve as intersection of these surfaces} \\ \varphi_1(x, y, z) = k_1 \quad \text{and} \quad \varphi_2(x, y, z) = k_2$$

if we take smooth fn $G(k_1, k_2)$ then we get integral surface

$$G(\varphi_1(x, y, z), \varphi_2(x, y, z)) = 0$$

we have the soln of this eqn $Ax + By = C$

ex Solve $y z_x - x z_y = 0$

$$A = y, \quad B = -x, \quad C = 0 \quad \Rightarrow \quad \frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0} \quad \text{note } \frac{dz}{0} \text{ means } z = C \text{ is the first integral / constant}$$

$$\varphi_1 \quad z = c_1$$

$$\varphi_2 \quad \frac{dx}{y} = -\frac{dy}{x} \quad \Rightarrow \quad x dx = -y dy \quad (\text{separable})$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c_2$$

so $\varphi_2: x^2 + y^2 = \tilde{c}_2$ 2nd first integral

general soln $G(x^2 + y^2, z) = 0$ we can solve for z by implicit fn then

$z = g(x^2 + y^2)$ where g is arbitrary smooth fn

ex solve $x^2 p + y^2 q - (x+y)z = 0$ Remark ($z_x = p$, $z_y = q$ is notation)

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1 \Rightarrow \varphi_1: \frac{1}{y} - \frac{1}{x} = c_1 \quad \text{first integral}$$

$$\frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad \text{Recall} \quad \frac{a}{b} = \frac{c}{d} \quad \text{then} \quad \frac{a}{b} = \frac{a+\mu c}{b+\mu d}$$

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx-dy}{(x-y)(x+y)} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx-dy}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \int \frac{d(x-y)}{x-y} = \int \frac{dz}{z} \quad \text{separable eqn}$$

$$= \ln|x-y| = \ln|z| + c_2$$

$$= x-y = z c_2 \quad (\text{we have new } c_2)$$

$$\Rightarrow \frac{x-y}{z} = c_2$$

so, general soln $G(c_1, c_2) = 0$ that is

$$G\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0 \quad \text{we can solve for } z$$

$$\frac{x-y}{z} = f\left(\frac{1}{y} - \frac{1}{x}\right) \quad \text{OR} \quad z = (x-y) \tilde{f}\left(\frac{x-y}{x-y}\right)$$

ex Solve $(x+z)z_x + yz_y = z + y^2$

$$\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$$

$$\varphi_1: \frac{dy}{y} = \frac{dz}{z+y^2} \Rightarrow \frac{dz}{dy} = \frac{1}{y} z + y^2 : \text{linear ODE for } z(y)$$

$$z' - \frac{1}{y} z = y^2 \quad \text{integrating factor } \mu : \frac{d\mu}{\mu} = -\frac{1}{y} dy \quad (C=0)$$

$$\frac{1}{y} z' - \frac{1}{y^2} z = 1$$

$$\ln|\mu| = \ln|y^{-1}| \quad \text{so } \mu = \frac{1}{y}$$

$$\int \left(\frac{1}{y} z\right)' = 1 \Rightarrow \frac{1}{y} z = y + c_2 \Rightarrow \boxed{\varphi_1: \frac{1}{y} z - y = c_2}$$

$$\varphi_2: \frac{dx}{x+z} = \frac{dy}{y} \quad \text{and use } \varphi_1: \frac{1}{y} z - y = c_1 \Rightarrow z = y^2 + c_1 y$$

$$\frac{dx}{x+y^2+c_1 y} = \frac{dy}{y} \Rightarrow \frac{dx}{dy} = \frac{x}{y} + y + c_1 \quad c_1 = \frac{1}{y} z - y$$

$$x' - \frac{1}{y} x = y + c_1 \quad \text{we know } \mu = \frac{1}{y}$$

$$\int \left(\frac{1}{y} x \right)' = \int 1 + \frac{c_1}{y} \quad \text{and we get } \Rightarrow \frac{1}{y} x = y + c_1 \ln y + c_2$$

$$\varphi_2: \frac{x}{y} - y - \underbrace{\left(\frac{1}{y} z - y \right)}_{c_1} \ln y = c_2$$

$$\text{So general soln is } G(c_1, c_2) = 0 \quad \text{or } G\left(\frac{1}{y} z - y, \frac{x}{y} - y - \left(\frac{1}{y} z - y\right) \ln y\right) = 0$$

application ex

given a family of surfaces $z^2 = r \cdot xy$ find all surfaces that are orthogonal to the given family



recall two surfaces are orthogonal at an intersection pt if their normals are orthogonal

$$z^2 = cxy \Rightarrow \frac{z^2}{xy} = c \quad \text{recall } (F(x, y, z)) = c, \quad N = (F_x, F_y, F_z)$$

$$\vec{N} = \left(-\frac{z^2}{x^2 y}, -\frac{z^2}{x y^2}, \frac{2z}{xy} \right)$$

$$\frac{\partial}{\partial x} \left(\frac{z^2}{y} \cdot \frac{1}{x} \right)$$

take $z = z(x, y)$ it has normal $(z_x, z_y, -1)$

$$z(x, y) - z = 0 \quad \text{we need } N_1 \perp N_2 : \vec{N}_1 \cdot \vec{N}_2 = 0$$

$$-\frac{z^2}{x^2 y} z_x - \frac{z^2}{x y^2} z_y - \frac{2z}{xy} = 0 \quad \text{now solve the eqn}$$

$$\frac{dx}{-\frac{z^2}{x^2 y}} = \frac{dy}{-\frac{z^2}{x y^2}} = \frac{dz}{\frac{2z}{xy}}$$

$$\text{Now } \varphi_1: \frac{-x^2 y dx}{z^2} = \frac{xy^2 dy}{z^2}$$

$$\int x dx = \int y dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\varphi_1: x^2 - y^2 = c_1$$

$$-\frac{x^2 y dx}{z^2} = \frac{-xy^2 dy}{z^2} = \frac{xy dz}{2z}$$

$$\varphi_2: \frac{-x^2 y}{z^2} dx = \frac{xy dz}{2z}$$

$$\int -x dx = \int \frac{z}{2} dz = -\frac{1}{2} x^2 = \frac{1}{4} z^2 + c_2$$

$$\varphi_2: z^2 + 2x^2 = c_2 \quad \text{so the general soln is}$$

↳ new c_2

$$G(\varphi_1, \varphi_2) = 0$$

$$G(x^2 - y^2, z^2 + 2x^2) = 0 \quad \text{solve wrt second one}$$

$$z^2 + 2x^2 = g(x^2 - y^2) \quad \text{OR} \quad z^2 = -2x^2 + g(x^2 - y^2)$$

let us justify our method of soln

Thm consider a quasilinear diff eqn

$$A(x, y, z) z_x + B(x, y, z) z_y = C(x, y, z) \quad \text{where } (x, y, z) \in \Omega$$

we assume $A, B, C \in C^1(\Omega)$ and $A^2 + B^2 \neq 0$ in Ω

if $\varphi_1(x, y, z) = c_1$ and $\varphi_2(x, y, z) = c_2$ are functionally independent,

first integrals of the system

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \quad (=dt) \quad \text{then for any continuously differentiable fn } G, \text{ the equality}$$

$G(c_1, c_2) = 0$ that is $G(\varphi_1(x, y, z), \varphi_2(x, y, z)) = 0$ defines a soln of the

given eqn. We assume that $G(c_1, c_2) = 0$ has solutions

in MT we may be asked to formulate a **Thm**

functionally dependent $\varphi_1(t) = \varphi_2$? Recall φ_1, φ_2 fn independent if $\varphi_1 \neq f(\varphi_2)$

$\varphi_1(x, y, z), \varphi_2(x, y, z)$ are functionally indep if $\nabla \varphi_1$ and $\nabla \varphi_2$ are linearly indep

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Recall we are solving quasi linear PDE

ex Solve $xz z_x - yz z_y = e^z$

consider the following system

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{e^z} \Rightarrow \frac{dx}{x} = \frac{dy}{-y}$$

$$\Rightarrow \int \frac{dx}{x} = \int -\frac{dy}{y}$$

$$= \ln x = -\ln y + c_1$$

$$\ln xy = c_1 \Rightarrow \varphi_1: xy = e^{c_1} = c_1$$

now $\frac{dx}{xz} = \frac{dz}{e^z}$ OR $\frac{dy}{-yz} = \frac{dz}{e^z}$

$\Downarrow$

$$\int \frac{dx}{x} = \int ze^{-z} dz$$

$$\ln x = -ze^{-z} - e^{-z} + c$$

$$\varphi_2: \ln x + ze^{-z} - e^{-z} = c_2$$

$$\Rightarrow \frac{dy}{-y} = ze^{-z} \Rightarrow -\ln y = -ze^{-z} - e^{-z} + c$$

$$\tilde{\varphi}_2: -\ln y + ze^{-z} + e^{-z} = c_2$$

$\varphi_1, \varphi_2, \tilde{\varphi}_2$ are functionally related

using φ_1, φ_2 we get $\tilde{\varphi}_2$

$$\ln \varphi_1 = \ln x + \ln y$$

$$\ln x = \ln \varphi_1 - \ln y$$

$$\varphi_2: \ln x + ze^{-z} + e^{-z}$$

$$\varphi_2: \ln \varphi_1 - \ln y + ze^{-z} + e^{-z}$$

$$\tilde{\varphi}_2: \varphi_2 - \ln \varphi_1$$

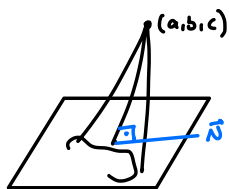
returning to the question, φ_1, φ_2 are enough

general soln: $F(\varphi_1, \varphi_2) = 0$ OR

$$F(xy, \ln x + ze^{-z} + e^{-z}) = 0$$

where F any smooth fn

ex find an first order PDE satisfied by conical surfaces with vertex at (a, b, c)



let $z = z(x, y)$ be such soln

take a pt (x, y, z) then the normal at (x, y, z) is $(z_x, z_y, -1)$

the Normal $\vec{N}$ is orthogonal to line through (a, b, c) and (x, y, z) (the line lies on the surface)

so the vectors $(z_x, z_y, -1)$ and $(a-x, b-y, c-z)$ are orthogonal, their inner product = 0

$$\langle (z_x, z_y, -1), (a-x, b-y, c-z) \rangle = 0$$

$$(a-x)z_x + (b-y)z_y - (c-z) = 0 \quad \text{let's solve it}$$

$$\frac{dx}{a-x} = \frac{dy}{b-y} = \frac{dz}{c-z}$$

$$\frac{dx}{a-x} = \frac{dy}{b-y} \Rightarrow -\ln|a-x| = -\ln|b-y| + c_1 \quad \varphi_1: \frac{b-y}{a-x} = c_1 \quad \text{first integral}$$

$$\frac{dx}{a-x} = \frac{dz}{c-z} \Rightarrow -\ln|a-x| = -\ln|c-z| + c_2 \quad \varphi_2: \frac{c-z}{a-x} = c_2$$

the soln $F\left(\frac{b-y}{a-x}, \frac{c-z}{a-x}\right) = 0$ for any smooth function F

Thm consider a quasi linear PDE

$$A(x, y, z)z_x + B(x, y, z)z_y = C(x, y, z) \quad \text{in } \Omega$$

we assume A, B, C are smooth in Ω and $A^2 + B^2 + C^2 \neq 0$ in Ω

if φ, ψ are first integrals of $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}$

then for any smooth F , the equality $F(\varphi, \psi) = 0$ defines a soln

we assume $F(\varphi, \psi) = 0$ can be solved in Ω

pf we have $z(x, y)$ is implicitly defined by

$$F(\varphi_1(x, y, z), \varphi_2(x, y, z)) = 0 \quad \text{we diff wrt } x \text{ and } y$$

remark for $F(t_1, t_2)$ we set $F_1 = \frac{\partial}{\partial t_1} F$, $F_2 = \frac{\partial}{\partial t_2} F$

$$F_1(\varphi_x + \varphi_z \cdot z_x) + F_2(\varphi_x + \varphi_z z_x) = 0$$

$$F_1(\varphi_y + \varphi_z \cdot z_y) + F_2(\varphi_y + \varphi_z z_y) = 0$$

$$\text{it follows} \quad -\frac{F_2}{F_1} = \frac{\varphi_x + \varphi_z z_x}{\varphi_x + \varphi_z z_x} = \frac{\varphi_y + \varphi_z z_y}{\varphi_y + \varphi_z z_y}$$

$$\Rightarrow (\varphi_y \varphi_z - \varphi_z \varphi_y) z_x + (\varphi_z \varphi_x - \varphi_x \varphi_z) z_y = \varphi_x \varphi_y - \varphi_y \varphi_x \quad (*)$$

satisfied by $z = z(x, y)$

we know on soln of the system

$$\varphi(x, y, z) = c_1 \quad \text{and} \quad \psi(x, y, z) = c_2 \quad . \quad \text{we diff the above equalities wrt } t$$

$$\varphi_x x' + \varphi_y y' + \varphi_z z' = 0$$

$$\text{or} \quad \frac{dx}{dt} = A \quad \frac{dy}{dt} = B \quad \frac{dz}{dt} = C$$

$$\varphi_x x' + \varphi_y y' + \varphi_z z' = 0$$

$$\Rightarrow \begin{cases} \varphi_x A + \varphi_y B + \varphi_z C = 0 \\ \psi_x A + \psi_y B + \psi_z C = 0 \end{cases}$$

$$C \neq 0 \quad \varphi_x \frac{A}{C} + \varphi_y \frac{B}{C} = -\varphi_z$$

$$\varphi_x \frac{A}{C} + \varphi_y \frac{B}{C} = -\varphi_z$$

$$\frac{B}{C} = -\frac{\varphi_x \varphi_z - \varphi_z \varphi_x}{\varphi_y \varphi_x - \varphi_x \varphi_y}$$

$$\frac{A}{C} = -\frac{\varphi_z \varphi_y - \varphi_y \varphi_z}{\varphi_y \varphi_x - \varphi_x \varphi_y}$$

compare with (*) we get

$$\frac{A}{C} z_x + \frac{B}{C} z_y = 1 \quad \text{hence} \quad A z_x + B z_y = C$$

MTA
statement of Thm
which proves F is soln

we consider the case of 2 indep. variables, it can be generalized to n -indep variables

Then consider the following eqn

$$P_1(x_1, \dots, x_n, z)z_{x_1} + P_2(x_1, \dots, x_n, z)z_{x_2} + \dots + P_n(x_1, \dots, x_n, z)z_{x_n} = Q(x_1, \dots, x_n, z)$$

in $\Omega \subset \mathbb{R}^{n+1}$ Assume $P_1, \dots, P_n, Q$ are smooth in Ω and $P_1^2 + \dots + P_n^2 + Q^2 \neq 0$

if $\varphi_1, \dots, \varphi_n$ are functionally indep, first integrals are

$$\frac{dx_1}{P_1} = \dots = \frac{dx_n}{P_n} = \frac{dz}{Q}$$

then for any smooth F the equality $F(\varphi_1, \varphi_2, \dots, \varphi_n) = 0$ defines a soln for PDE

ex solve $xu_x + 2yuy_z + 3zuz = 4u$

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{3z} = \frac{du}{4u} (= dt)$$

$$\frac{dx}{x} = \frac{dy}{2y} \Rightarrow \ln|x| = \frac{1}{2} \ln|y| + C \Rightarrow \frac{x^2}{y} = c_1 : \varphi_1 : \text{first integral}$$

$$\frac{dy}{2y} = \frac{dz}{3z} \Rightarrow \frac{1}{2} \ln|y| = \frac{1}{3} \ln|z| + C \Rightarrow \frac{y^3}{z^2} = c_2 : \varphi_2 : \text{first integral}$$

$$\frac{dx}{x} = \frac{dz}{3z} \Rightarrow \text{DONT TAKE it's not fn indep}$$

$$\frac{dx}{x} = \frac{du}{4u} \Rightarrow \ln|x| = \frac{1}{4} \ln|u| \Rightarrow \frac{u}{x^4} = c_3 : \varphi_3 : \text{first integrals}$$

for any smooth $F\left(\frac{x^2}{y}, \frac{y^3}{z^2}, \frac{u}{x^4}\right) = 0$ is a soln

we can solve wrt u

$$\frac{u}{x^4} = f\left(\frac{x^2}{y}, \frac{y^3}{z^2}\right) \Rightarrow u = x^4 f\left(\frac{x^2}{y}, \frac{y^3}{z^2}\right)$$

ex Solve $(y-z)u_x + (z-x)u_y + (x-y)u_z = 0$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0} = dt$$

$$\frac{du}{0} \Leftrightarrow \frac{du}{dt} = 0 \quad u = c_1 \quad \text{first integral}$$

$$\frac{dx+dy+dz}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0} = dt \Rightarrow d(x+y+z) = 0 \Rightarrow x+y+z = c_2 : \text{first integral}$$

$$\frac{x dx + y dy + z dz}{x(y-z) + y(z-x) + z(x-y)} = dt \Rightarrow \int \frac{x dx + y dy + z dz}{0} = \int dt \Leftrightarrow \frac{d(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2})}{0} = dt$$

$$\Rightarrow \frac{x^2 + y^2 + z^2}{2} = c$$

$$F(x+y+z, x^2+y^2+z^2, u) = 0$$

$$u = f(x+y+z, x^2+y^2+z^2) \text{ for any } f \text{ smooth}$$

$$\text{or } x^2+y^2+z^2 = c_3 : \text{first integral}$$

Cauchy Problem for 1st order quasi linear PDE

first we consider the case of two indep variables $z = z(x, y)$

find a soln of $A(x, y, z)z_x + B(x, y, z)z_y = C(x, y, z)$ passing through a given curve

$$x = \alpha(s) \quad y = \beta(s) \quad z = \gamma(s) \quad s \in (a, b)$$

idea we construct solution surfaces using characteristic curves. Take and through each

pt of Γ we pass a characteristic. That is we solve



$$\frac{dx}{dt} = A \quad x(0) = \alpha(s)$$

$$\frac{dy}{dt} = B \quad y(0) = \beta(s)$$

$$\frac{dz}{dt} = C \quad z(0) = \gamma(s)$$

In general, we get a soln

$$\left. \begin{array}{l} x = x(t, s) \\ y = y(t, s) \\ z = z(t, s) \end{array} \right\} \text{Next we want to write } z \text{ is a fn of } x \text{ and } y$$

We solve t, s in terms of x and y from $x = x(t, s)$ and $y = y(t, s)$ and substitute in

$z = z(t, s) = z = z(x, y)$ in some neighborhood of Γ

2.7 Subst Pers

note we can solve t, s from

$$\begin{aligned} x &= x(t, s) \\ y &= y(t, s) \end{aligned} \quad \text{if} \quad \begin{vmatrix} x_t & y_t \\ x_s & y_s \end{vmatrix} \neq 0 \quad \text{on } \Gamma$$

ex Solve Cauchy $u u_x + u_y = 1$ $u(x, x) = 0$

$$u(x, x) = 0 \Leftrightarrow x = s, y = s, u = 0.$$

the system for char.

$$\begin{cases} \frac{dx}{dt} = u & x(0) = s \\ \frac{dy}{dt} = 1 & y(0) = s \\ \frac{dz}{dt} = 1 & z(0) = 1 \end{cases} \quad \begin{aligned} \frac{dy}{dt} = 1 & \Rightarrow \int dy = \int dt = y = t + C \quad y(0) = s \Rightarrow C = s \\ & y = t + s \\ \frac{dz}{dt} = 1 & \Rightarrow \int dz = \int dt \Rightarrow z = t + C \quad z(0) = 1 \Rightarrow C = 0 \\ & z = t \\ \frac{dx}{dt} = t & \Rightarrow \int dx = \int t dt \Rightarrow x = \frac{t^2}{2} + C \quad x(0) = s \Rightarrow C = s \\ & x = \frac{t^2}{2} + s \end{aligned}$$

$$u = u(x, y)$$

$$\begin{cases} x = \frac{1}{2} t^2 + s \\ y = t + s \\ u = t \end{cases} \quad \left. \begin{aligned} x - y &= \frac{1}{2} t^2 - t \\ t^2 - 2t + 2(y - x) &= 0 \end{aligned} \right\}$$

$$t = \frac{2 \pm \sqrt{4 - 8(y - x)}}{2} = 1 \pm \sqrt{1 - 2(y - x)}$$

$$\text{so, } u = \sqrt{1 - 2(y - x)} \quad u|_{\Gamma} = 0 \quad x = y$$

Recall for $A x_x + B x_y = C$ we found general soln by using first integral

Suppose $\psi = c_1$, $\varphi = c_2$ are the first integral and general soln

$$F(\psi, \varphi) = 0, \quad F \text{ any smooth fn. Can we use } F(\psi, \varphi) = 0?$$

we need to find F st points of Γ satisfy $F(\psi, \varphi) = 0$

we need to find a relation between $\psi = c_1$ and $\varphi = c_2$ satisfied on Γ

ex for the eqn $xz z_x + yz z_y = -xy$

a) find the general soln

b) find soln passing through the curve $y = x^2$ $z = x^2$ ($x=s$, $y=s^2$, $z=s^2$)

$$\underline{a)} \quad \frac{dx}{xz} = \frac{dz}{yz} = \frac{dy}{xy} \quad , \quad \frac{dx}{xz} = \frac{dy}{xy} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \ln|x| = \ln|y| + C$$

$$\ln\left|\frac{x}{y}\right| = C$$

$$\frac{dx}{xz} = -\frac{dz}{xy} \Rightarrow y dx = -z dz$$

$$\frac{x}{y} = C_1 \quad \text{first integral}$$

$$\varphi_1: \frac{x}{y} = C_1$$

$$\text{from } \varphi_1: \frac{x}{y} = C_1 \Rightarrow y = \frac{x}{C_1}$$

$$\int \frac{x}{C_1} dx = \int -z dz \Rightarrow \frac{x^2}{2C_1} = -\frac{z^2}{2} + C \Rightarrow \frac{x^2}{C_1} = z^2 = C \quad \left(\frac{x}{y}\right) + z^2 = C$$

$$\text{gen soln} \quad F\left(\frac{x}{y}, xy + z^2\right) = 0$$

b) $\Gamma \quad y = x^2, \quad z = x^3$

$$\frac{x}{y} = C_1, \quad xy + z^2 = C_2$$

$$\text{if } \frac{1}{x} = C_1, \quad x^3 + x^6 = C_2$$

$$x = \frac{1}{C_1} \quad \left(\frac{1}{C_1}\right)^3 + \left(\frac{1}{C_1}\right)^6 = C_2 \quad \text{relation satisfied on } \Gamma$$

$$C_1^3 + 1 - C_1^6 C_2 = 0$$

$$\text{the soln} \quad \left(\frac{x}{y}\right)^3 + 1 - \left(\frac{x}{y}\right)^6 (xy + z^2) = 0 \quad \text{we have } z \text{ as implicit fn}$$

ex find the soln for $x(y^2+z)z_x - y(x^2+z)z_y = (x^2-y^2)z$

passing through Γ $x=t$ $y=-t$ $z=1$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \Rightarrow \frac{x dx + y dy}{x^2 y^2 + x^2 z - y^2 x^2 - y^2 z} = \frac{dz}{(x-y)(x+y)z}$$

$$\frac{d\left(\frac{x^2}{2} + \frac{y^2}{2}\right)}{z(x^2-y^2)} = \frac{dz}{(x-y)(x+y)z}$$

$$\frac{1}{2} dx^2 + y^2 = dz$$

$$d\left(\frac{x^2+y^2}{2}, z\right) = 0 \quad \frac{x^2+y^2}{2} - z = C \Rightarrow \varphi: x^2+y^2-2z = C_1$$

note $d(x^2+y^2) = 2x dx + 2y dy$

$$d(x^2-y^2) = 2x dx - 2y dy$$

$$d(xy) = y dx + x dy$$

sometimes $d(x^3+y^3)$ or $d(x^5-y^5)$

$$\frac{y dx + x dy}{y(x(y^2+z)) + x(-y(x^2+z))} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{dxy}{xy^3 + xyz - yx^3 - xyx} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{dxy}{xy(y^2-x^2)} = \frac{dz}{(x^2-y^2)} \quad , \quad -\frac{dxy}{xy} = \frac{dz}{z}$$

$$-\int \frac{dxy}{xy} = \int \frac{dz}{z} \Rightarrow -\ln|xy| = \ln|z| + C$$

$$\ln|xyz| = C \quad \psi: xyz = C_2$$

general soln $F(x^2+y^2-2z, xyz) = 0$

relate $x^2+y^2-2z = C_1$, $xyz = C_2$ on Γ $x=t$ $y=-t$ $z=1$

$$2t^2 - 2 = C_1 \quad -2C_2 - 2 = C_1$$

$$-t^2 = C_2 \quad C_1 + 2C_2 + 2 = 0$$

$$F(C_1, C_2)$$

The soln is $2(xyz) + (x^2+y^2-2z) + 2 = 0$

Thm (Existence and Uniqueness)

consider $A(x, y, z) z_x + B(x, y, z) z_y = C(x, y, z) \quad x, y, z \in \Omega$

assume that $A, B, C \in C^1(\Omega) \quad A^2 + B^2 + C^2 \neq 0 \quad \text{in } \Omega$

let $\Gamma: x = \alpha(s), y = \beta(s), z = \gamma(s) \quad s \in (a, b)$ be a smooth curve

in Ω that is $\alpha, \beta, \gamma \in C^1(a, b)$

if $\Delta = \begin{vmatrix} A & B \\ \alpha' & \beta' \end{vmatrix} \neq 0$ on Γ then the above Cauchy problem has a unique soln in some neighborhood of Γ

pf consider the system for char.

$$\begin{cases} \frac{dx}{dt} = A & x(0) = \alpha(s) \\ \frac{dy}{dt} = B & y(0) = \beta(s) \\ \frac{dz}{dt} = C & z(0) = \gamma(s) \end{cases} \quad \begin{array}{l} \text{our assumptions on } A, B, C \text{ and } \alpha, \beta, \gamma \\ \text{guarantee that the system has a unique soln} \\ x = x(t, s) \quad y = y(t, s) \quad z = z(t, s) \end{array}$$

Since $\Delta|_{\Gamma} \neq 0$ from $x = x(t, s)$ and $y = y(t, s)$ we can find

$$t = t(x, y), \quad s = s(x, y)$$

Thus we have a soln $z = z(t(x, y), s(x, y))$

The uniqueness follows from

- 1) for each pt of Γ there is a unique charact. passing through the pt
- 2) If a charact. has a common pt with a soln surface it completely lies on that surface

4 Mart Solu

we consider the Cauchy problem to the first order quasi linear PDE

ex find the soln of $2 \cdot z_x - xy z_y = 2xz$ satisfying $x+y=2$ and $yz=1$

first integrals ?

$$\frac{dx}{z} = \frac{dz}{-xy} = \frac{dz}{2xz}$$


$$\frac{dx}{z} = \frac{dz}{2xz} \Rightarrow \int 2x dx = \int \frac{dz}{z} \Rightarrow x^2 = z + C$$

$$x^2 - z = C_1 \quad (\text{first integral})$$

$$\frac{dz}{-xy} = \frac{dz}{2xz} \Rightarrow 2 \int \frac{dz}{z} = \int \frac{dz}{-y} \Rightarrow -2 \ln y = \ln z + C$$

$$\ln y^2 + \ln z = -C \Rightarrow y^2 z = C_2 \quad (\text{first integral})$$

general soln $F(x^2 - z, y^2 z) = 0$ for F smooth

relate C_1, C_2 on the initial curve

$$\begin{aligned} x+y &= 2 & x &= 2-y & (2-y)^2 - \frac{1}{y} &= C_1 \\ yz &= 1 & z &= \frac{1}{y} & y &= C_2 \end{aligned} \Rightarrow (2-C_2)^2 - \frac{1}{C_2} = C_1$$

$$\text{so the soln is } (2 - y^2 z)^2 - \frac{1}{y^2 z} = x^2 - z \quad (\text{implicit fn})$$

Remark to have $z = z(x, y)$ in some nbh of Γ we need implicit fn thm on Γ

Recall we can solve Cauchy problem uniquely

$$A(x, y, z) z_x + B(x, y, z) z_y = C(x, y, z)$$

$$\Gamma: \quad x = x(s) \quad y = y(s) \quad z = z(s) \quad s \in (\alpha, \beta)$$

$$\text{If } \Delta|_{\Gamma} = \begin{vmatrix} A & B \\ x'(s) & y'(s) \end{vmatrix} \neq 0$$

we assume A, B, C and $x(s), y(s), z(s)$ satisfy

cond of the Existence thm

A, B, C are not zero simultaneously

$A^2 + B^2 + C^2 \neq 0$ $\vec{0}$ is orthogonal to any vector

What if $\Delta I_\Gamma = 0$

one can prove that if $\Delta I_\Gamma = 0$ then

- we have infinitely many solns if Γ is charact. curve
- we have no soln if Γ is NOT charact. curve

ex find soln for the Cauchy problem

$$xz_y - yz_x = 0 \quad \text{and} \quad x^2 + y^2 = 4, \quad z = 1$$

first integrals

$$\frac{dx}{-y} = \frac{dz}{x} = \frac{dz}{0} \quad \begin{matrix} \nearrow z = C_1 \text{ (symbolic rotation)} \\ z = C_1 \text{ is the first integral} \end{matrix}$$

$$\frac{dx}{-y} = \frac{dz}{x} \Rightarrow \int x dx = \int -y dy$$
$$\frac{x^2}{2} = \frac{-y^2}{2} + C \Rightarrow x^2 + y^2 = C_2$$

we have first integrals

$$z = C_1$$
$$x^2 + y^2 = C_2 \quad \text{Obviously } x^2 + y^2 = 4 \quad \text{and } z = 1 \text{ is a charact curve}$$

general soln $F(z, x^2 + y^2) = 0$ we can take any F st $F(1, 4) = 0$

$$F(z, x^2 + y^2) = 0 \quad z = f(x^2 + y^2) \quad \text{we can take any } f \text{ st } f(4) = 1$$

$$\text{ie } z = \frac{x^2 + y^2}{4} \quad \text{or } z = 5 - (x^2 + y^2) \quad \text{or } z = \cos(x^2 + y^2 - 4)$$

Since initial curve is charact we have inf many solns

why we have inf many solns when initial curve is charact.

$$Az_x + Bz_y = C \quad \text{and} \quad x = x(s) \quad y = y(s) \quad z = z(s) \quad \text{To construct a soln we take } \Gamma$$

and pass charact. through each pt of Γ

In such a way we get only Γ but we can take any non charact curve Γ intersecting Γ (Nice curve :)

and consider the Cauchy with Γ get soln



The obtained soln contains Γ so it also solve the original Cauchy Problem

ex determine if the following cauchy problem has unique/no/inf many solns

$$(y^2 - z)z_x + yz_y = z$$

note: No need to solve the problem

a) $\Gamma: x=1, y=s, z=\frac{s^2}{2}$

check $\Delta|_{\Gamma}$

b) $\Gamma: x=\frac{1}{2}s^2-s, y=s, z=s, s(2,3)$

$$a) \Delta|_{\Gamma} = \begin{vmatrix} A & B \\ x'(s) & y'(s) \end{vmatrix} = \begin{vmatrix} y^2 z & y \\ 0 & 1 \end{vmatrix} \Big|_{\Gamma} = \begin{vmatrix} s^2 \cdot \frac{s^2}{2} & s \\ 0 & 1 \end{vmatrix} = \frac{s^2}{2} - s$$

$\frac{s^2}{2} \neq 0$

since $\Delta|_{\Gamma} \neq 0$ we have a unique soln

$$b) \Delta|_{\Gamma} = \begin{vmatrix} A & B \\ x'(s) & y'(s) \end{vmatrix} = \begin{vmatrix} y^2 z & y \\ s-1 & 1 \end{vmatrix} \Big|_{\Gamma} = \begin{vmatrix} s^2-s & s \\ s-1 & 1 \end{vmatrix} = s^2-s - s^2-s = 0$$

$\Delta|_{\Gamma} = 0$ we have two cases : inf many or no soln check if Γ is charact.

$$\tau_{\Gamma} // (A, B, C)$$

↳ tangent vector of Γ

$$\tau_{\Gamma} = (x'(s), y'(s), z'(s)) = (s-1, 1, 1)$$

$$(A, B, C)|_{\Gamma} = (y^2 z, y, z)|_{\Gamma} = (s^2-s, s, s)$$

$$s \cdot \tau_{\Gamma} = (A, B, C)$$

so they are parallel
it not need to be constant

hence, Γ is charact we have inf many solns.

Pfaffian PDE

$Az_x + Bz_y = C$ We construct surfaces tangent to the vector field $\vec{F} = A\vec{i} + B\vec{j} + C\vec{k}$

consider another problem find surfaces orthogonal to a given vector field

$$\vec{G} = P\vec{i} + Q\vec{j} + R\vec{k} \quad \begin{matrix} (\vec{N} \perp \vec{F}) \\ N \cdot F = 0 \end{matrix} \quad \begin{matrix} (\vec{N} \parallel \vec{G}) \\ N = \lambda G \end{matrix}$$

normal to the soln surface parallel to G

any vector in tangent space to the soln surface (dx, dy, dz) is orthogonal to $\vec{G}$

$$\text{that is } (dx, dy, dz) \cdot (P, Q, R) = 0$$

$$Pdx + Qdy + Rdz = 0 \quad (\text{Pfaffian PDE})$$

ex suppose $\vec{G}$ is conservative vector fields

$$\vec{G} = \nabla \phi(x, y, z) \quad \phi \text{ is called potential of } G$$

$$G = \phi_x \vec{i} + \phi_y \vec{j} + \phi_z \vec{k}$$

then the level surfaces $\phi(x, y, z) = C$ will be normal to G . Indeed, $\vec{G} = N_\phi$ at each pt

Recall if we consider $\vec{G}$ in a simply connected domain Ω then $\vec{G}$ is conservative iff

$$\text{curl}(G) = 0$$

$$\text{curl}(G) = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k} = 0$$

ex determine if $G = (yz+1)\vec{i} + (xz+1)\vec{j} + (yx+1)\vec{k}$ is potential

$$\text{and solve } (yz+1)dx + (xz+1)dy + (yx+1)dz = 0 \quad \Omega = \mathbb{R}^3$$

$$\text{curl}(G) \stackrel{?}{=} 0$$

$$\text{curl}(G) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+1 & xz+1 & yx+1 \end{vmatrix} = (x-x)\vec{i} - (y-y)\vec{j} + (z-z)\vec{k} = 0$$

Since G is conservative ($G = P\vec{i} + Q\vec{j} + R\vec{k}$) soln of

$$Pdx + Qdy + Rdz = 0 \quad \text{is given by potential}$$

$$P = \phi_x \quad Q = \phi_y \quad R = \phi_z$$

$$\phi_x = yz+1 \Rightarrow \phi = yzx + x + C_1(y, z)$$

$$\phi_y = xz+1 \Rightarrow \frac{\partial}{\partial y} \phi = xz + \frac{\partial}{\partial y} C_1 = xz+1 \Rightarrow \frac{\partial}{\partial y} C_1 = 1 \quad C_1 = y + C_2(z)$$

$$\phi_z = xy+1 \Rightarrow \phi = xyz + x + y + C_2 \quad \text{(Rewrite)}$$

$$\frac{\partial}{\partial z} \phi = xy+1 \Rightarrow xy + \frac{\partial}{\partial z} C_2 = xy+1 \quad \frac{\partial}{\partial z} C_2 = 1 \quad C_2 = z + C \quad \text{take } C=0 \text{ need a potential}$$

$$\phi = xyz + x + y + z$$

the soln surfaces are $\phi = C : xyz + x + y + z = C$

if $G = \nabla \phi$ Pfaffian PDE has a soln

an other case when soln exist if $G \parallel \nabla \phi$

that is $\vec{G} = \mu(x, y, z) \cdot \nabla \phi$. Again $\phi = C$ would be a soln

vector field orthogonal to a surface $\begin{cases} \text{either conservative} \\ \text{proportional to conservative} \end{cases}$

let us show that these are two only cases when soln of $Pdx + Qdy + Rdz = 0$ exists.

Indeed let $\phi(x, y, z) = C$ be a soln

$$\text{Then } \vec{N}_\phi \parallel \vec{G} \Rightarrow G = \mu N_\phi = \mu \nabla \phi$$

$$\text{when } \vec{G} = \mu \cdot \nabla \phi \text{ for some } \phi$$

necessary cond for $\vec{G} = \mu \nabla \phi$

assume that soln exists we write $\phi(x, y, z) = 0$ as $z = z(x, y)$ then

we have $dz = z_x dx + z_y dy$ and we substitute into

$$Pdx + Qdy + Rdz = 0$$

$$Pdx + Qdy + R(z_x dx + z_y dy) = 0$$

$$(P + Rz_x)dx + (Q + Rz_y)dy = 0 \quad dx, dy \text{ are indep}$$

$$\begin{cases} P + R z_x = 0 \\ Q + R z_y = 0 \end{cases} \Rightarrow \begin{aligned} z_x &= -\frac{P}{R} \\ z_y &= -\frac{Q}{R} \end{aligned}$$

we consider smooth fn so $z_{xy} = z_{yx}$

$$\left(\frac{P}{R}\right)_y = \left(\frac{Q}{R}\right)_x \quad P(x, y, z(x, y)), Q(x, y, z(x, y)), R(x, y, z(x, y))$$

$$\frac{R(P_y + P_z z_y) - P(R_y + R_z z_y)}{R^2} = \frac{R(Q_x + Q_z z_x) - Q(R_x + R_z z_x)}{R^2}$$

Nozmi Oyar Help Room Sat 14.40 - 16.30

6 Mart Cor

we consider the Pfaffion PDE

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \quad \text{in } \Omega \in \mathbb{R}^3$$

we know that soln exists if

$$P\vec{i} + Q\vec{j} + R\vec{k} = \mu(x, y, z) \nabla \Phi(x, y, z) \quad \text{for some } \Phi$$

Necessary cond for existence of soln

(assume soln exist
 $z = z(x, y)$)

subs to the eqn

$$Pdx + Qdy + R(z_x dx + z_y dy) = 0$$

$$(P + R z_x)dx + (Q + R z_y)dy = 0 \quad (dx, dy \text{ indep})$$

$$\Rightarrow \begin{aligned} P + R z_x &= 0 \\ Q + R z_y &= 0 \end{aligned}$$

$$\begin{cases} z_x = -\frac{P}{R} \\ z_y = -\frac{Q}{R} \end{cases} \quad z_{xy} = z_{yx} \Rightarrow \frac{\partial}{\partial y} \left(-\frac{P}{R}\right)_y = \frac{\partial}{\partial x} \left(-\frac{Q}{R}\right)_x$$

$$\Rightarrow P(x, y, z(x, y)), Q(x, y, z(x, y)), R(x, y, z(x, y)) \quad \text{so,}$$

$$\frac{(P_y + P_z z_y)R - P(R_y + R_z z_y)}{R^2} = \frac{(Q_x + Q_z z_x)R - Q(R_x + R_z z_x)}{R^2}$$

the equality

$$(p_y + \underbrace{p_z z_y}_{-\frac{\partial}{\partial z}}) R - P(\underbrace{R_y + \underbrace{R_z z_y}_{-\frac{\partial}{\partial z}}}) = (\underbrace{\partial_x + \underbrace{\partial_z z_x}_{-\frac{\partial}{\partial z}}}) R - \partial_z(R_x + \underbrace{R_z z_x}_{-\frac{\partial}{\partial z}})$$

$$p_y R - p_z \partial_z - P R_y + \cancel{\frac{\partial P p_z}{R}} = \partial_x R - \partial_z P - \partial_z R_x + \cancel{\frac{\partial P R_z}{R}}$$

$$P(\partial_z - R_y) + \partial_z(R_x - P_z) + R(p_y - \partial_x) = 0$$

we have $(P, \partial_z, R) \cdot \text{curl}(P, \partial_z, R) = 0$

the vector field $\vec{G} = P\vec{i} + \partial_z\vec{j} + R\vec{k}$ MUST SATISFY $\vec{G} \cdot \text{curl } \vec{G} = 0$

Remark we also can look for soln by solving

$$\begin{cases} z_x = -\frac{P}{R} \\ z_y = -\frac{\partial_z}{R} \end{cases}$$

ex Solve $y z dx + 2 x z dy + x y dz = 0 \quad x, y, z > 0$

first check necessary cond (if soln exist)

$\vec{G} = (yz)\vec{i} + (2xz)\vec{j} + (xy)\vec{k} \quad \vec{G} \cdot \text{curl } \vec{G} \stackrel{?}{=} 0$

$$\text{curl } \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & xy \end{vmatrix} = -x\vec{i} - (0)\vec{j} + z\vec{k} =$$

Curl is NOT zero

vector field to potential is not conservative ?? **Ask**

$\vec{G} \cdot \text{curl } \vec{G} = (yz, 2xz, xy) \cdot (-x, 0, z) = -xyz + xyz = 0$ soln exist (We'll prove that this soln $\vec{G} \cdot \text{curl } \vec{G} = 0$ is a sufficient cond)

$z_x = -\frac{yz}{xy} = -\frac{z}{x} \Rightarrow \int \frac{dz}{z} = \int -\frac{dx}{x} \Rightarrow \ln z = -\ln x + B \Rightarrow z = \frac{B}{x} \quad \text{so } z = \frac{B(y)}{x}$

$z_y = -\frac{2xz}{xy} = -\frac{2z}{y}$

$\frac{\partial}{\partial y} \left(\frac{B(y)}{x} \right) = -2 \frac{B'(y)}{xy}$

$\frac{B'}{x} = -\frac{2B}{xy} \Rightarrow \int B' = \int -\frac{2}{y} B$

$= \ln(B) = -2 \ln(y) + C \Rightarrow B = \frac{C}{y^2}$

Soln $\boxed{z = \frac{C}{xy^2}}$

remember $z = \frac{B}{x}$

ex Solve $y^2 z dx + 2xz dy + xy dz = 0$

first check if soln exist

$$\vec{G} = (y^2 z, 2xz, xy) \quad \text{curl } G = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & 2xz & xy \end{vmatrix} = (-x)\vec{i} - (y - y^2)\vec{j} + (2z - 2yz)\vec{k}$$

$$G \cdot \text{curl } G = (-x)y^2 z + (y^2 - y)2xz + (2z - 2yz)xy = -xy^2 z \neq 0 \quad \text{no soln exists}$$

necessary cond holds $\Rightarrow$ don't mean soln exists $\Rightarrow ?$

no necessary cond $\Rightarrow$ no soln

sufficient cond holds $\Rightarrow$ soln exist

no sufficient cond $\Rightarrow ?$

Thm let P, Q, R be cont diffble fns in some open simply connected Ω .

Then through each pt of Ω passes it exists exactly 1 surface $z = z(x, y)$ satisfying

$$Pdx + Qdy + Rdz = 0 \quad \text{iff the vector field } G \text{ equal to}$$

$$G = P\vec{i} + Q\vec{j} + R\vec{k} \quad \text{satisfy} \quad G \cdot \text{curl } G = 0$$

(also $P^2 + Q^2 + R^2 \neq 0$ in thm)

pf $\Rightarrow$ if soln exists then $G \cdot \text{curl } G = 0$ is proved

$\Leftarrow$ Suppose $G \cdot \text{curl } G = 0$

we know that it's enough to show

$$z_x = -\frac{P}{R}$$

$$\text{denote } -\frac{P}{R} = \bar{P}, \quad -\frac{Q}{R} = \bar{Q}$$

$$z_y = -\frac{Q}{R}$$

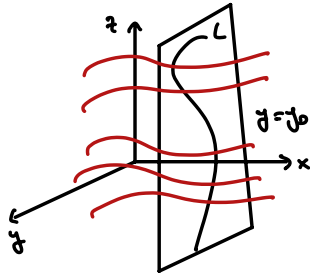
$$\text{we need to show } \begin{matrix} z_x = \bar{P} \\ z_y = \bar{Q} \end{matrix}$$

$$\text{also } G \cdot \text{curl } G = 0 \Leftrightarrow (\bar{P})_y = (\bar{Q})_x \quad (z_{xy} = z_{yx})$$

fix $(x_0, y_0, z_0) \in \Omega$

and consider $z_x = \bar{P}(x, y_0, z)$: ODE, $\bar{P}$ is smooth hence $\exists$ a unique soln

satisfying $z(x_0) = z_0$. We have a curve through that pt (x_0, y_0, z_0)



next for each pt (x_L, y_L, z_L)

we solve $z_y = \bar{Q}(x_L, y, z)$ $z(y_0) = z_L$

we get family of curves L_{x_L}

the union of curves L_{x_L} generates unique surface S passing thru (x_0, y_0, z_0)

let us show S is a soln

by construction $z_y = \bar{Q}$ is satisfied on S but for $z_x = \bar{P}$ is satisfied on L for $y = y_0$

let us prove that $z_x = P$ on S

consider $A = z_x - P$ ($A = 0$?) $z_x = A + \bar{P}$

we have $A_y = \underbrace{z_{xy}}_{z_{yx}} - \bar{P}_y - \bar{P}_z z_y = (\bar{Q}_x + \bar{Q}_z z_x) - (\bar{P}_y - \bar{P}_z z_y)$

$$A_y = \bar{Q}_x + \bar{Q}_z (A + \bar{P}) - (\bar{P}_y + \bar{P}_z \bar{Q})$$

$$A_y = \bar{Q}_x + \bar{Q}_z A + \bar{Q}_z \bar{P} - \bar{P}_y - \bar{P}_z \bar{Q}$$

$$(\bar{P})_y = (\bar{Q})_x \Leftrightarrow \bar{P}_y + \bar{P}_z \bar{Q} = \bar{Q}_x + \bar{Q}_z \bar{P}$$

$$A_y = \bar{Q}_x + \bar{Q}_z A + \bar{Q}_z \bar{P} - \bar{Q}_x - \bar{Q}_z \bar{P}$$

$$A_y = \bar{Q}_z A \Rightarrow A = C \cdot e^{\int_{y_0}^y \bar{Q}_z dy} \quad \text{but } z_x = \bar{P} \text{ for } y = y_0 \text{ so } A = 0 \quad y = y_0$$

$$\text{set } y = y_0 \quad D = C \cdot 1 \Rightarrow C = 0 \quad A = 0$$

we have $z_x = \bar{P}$ for all y $\square$

Remark considering $Pdx + Qdy + Rdz = 0$ we obtained necessary and sufficient conds.

for existence of soln but we also obtained necessary and sufficient conds for existence of soln for

$$\begin{cases} z_x = \bar{P}(x, y, z) \\ z_y = \bar{Q}(x, y, z) \end{cases} \quad \text{soln exist iff } \frac{\partial}{\partial y}(\bar{P}) = \frac{\partial}{\partial x}(\bar{Q}) \quad \text{that is } \bar{P}_y + \bar{P}_z \bar{Q} = \bar{Q}_x + \bar{Q}_z \bar{P}$$

ex for a system

$$z_x = \left(\frac{x}{x^2+y^2} + y \right) z \quad \text{a) determine if soln exist}$$

$$z_y = \left(\frac{y}{x^2+y^2} + x \right) z \quad \text{b) find soln (if exist)}$$

a) $z_{xy} = z_{yx}$

$$z_{xy} = \left(x \cdot \frac{-2y}{(x^2+y^2)^2} + 1 \right) z + \left(\frac{x}{x^2+y^2} + y \right) \cdot z_y$$

$$= \frac{-2xy z}{(x^2+y^2)^2} + xz + \left(\frac{x}{x^2+y^2} + y \right) \left(\frac{y}{x^2+y^2} + x \right) z$$

$$z_{yx} = \left(y \cdot \frac{-2x}{(x^2+y^2)^2} + 1 \right) z + \left(\frac{y}{x^2+y^2} + x \right) z_x$$

$$= \frac{-2xy z}{(x^2+y^2)^2} + z + \left(\frac{y}{x^2+y^2} + x \right) \left(\frac{x}{x^2+y^2} + y \right) z$$

$$z_{xy} = z_{yx} \quad \text{Soln exists}$$

b) $z_x = \left(\frac{x}{x^2+y^2} + y \right) z$

$$y: \text{fixed} \quad \frac{dz}{dx} = \left(\frac{x}{x^2+y^2} + y \right) z \quad \Rightarrow \quad \int \frac{dz}{z} = \int \left(\frac{x}{x^2+y^2} + y \right) dx$$

$$= \ln z = \frac{1}{2} \ln(x^2+y^2) + yx + B(y)$$

$$z = e^{\frac{1}{2} \ln(x^2+y^2)} \cdot B(y)$$

$$B=? \quad z_y = \left(\frac{y}{x^2+y^2} + x \right) z = \left(\frac{y}{x^2+y^2} + x \right) e^{\frac{1}{2} \ln(x^2+y^2)} B(y)$$

||

$$z_y = \left(e^{\frac{1}{2} \ln(x^2+y^2)} \cdot B(y) \right)_y = x e^{\frac{1}{2} \ln(x^2+y^2)} B(y) + e^{\frac{1}{2} \ln(x^2+y^2)} \cdot \frac{B'(y)}{2 \sqrt{x^2+y^2}} B(y) + e^{\frac{1}{2} \ln(x^2+y^2)} B'(y)$$

$$e^{\frac{1}{2} \ln(x^2+y^2)} \cdot B'(y) = 0 \quad \Rightarrow \quad B'(y) = 0 \quad B = C$$

$$z = C e^{\frac{1}{2} \ln(x^2+y^2)}$$

11 More Sol

ex a) $(2yz + 3x) dx + xz dy + xy dz = 0$

b) $(x-y) dx + z dy - x dz = 0$

determine if a soln exists and find soln (if exist)

a) existence: $G \cdot \text{curl } G = 0$, $G = (2yz + 3x)i + xzj + xyk$

$$\text{curl } G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz+3x & xz & xy \end{vmatrix} = 0i - (y-2y)j + (z-2z)k = yj - zk$$

$G \cdot \text{curl } G \Rightarrow y \cdot xz + (-z)xy = 0$ soln exist

find a soln for $(2yz + 3x) dx + xz dy + xy dz = 0$ $z?$ $dz = \dots$

$$dz = \frac{2yz}{xy} dx - \frac{xz}{xy} dy$$

$$z_x = -\frac{2yz + 3x}{xy} \quad \left(-\frac{p}{r}\right)$$

$$z_y = -\frac{xz}{xy} \quad \text{pick this, easy} \quad -\left(\frac{q}{r}\right)$$

$$z_y = -\frac{xz}{xy} \Rightarrow z_y = -\frac{z}{y}, \text{ solve } \frac{dz}{dy} = -\frac{z}{y}$$

$$\int \frac{dz}{z} = \int -\frac{dy}{y}$$

$$= \ln z = -\ln y + B(x)$$

$$zy = B(x)$$

$$z = \frac{B(x)}{y} \text{ now subst. in } z_x = -\frac{2yz + 3x}{xy}$$

$$\Rightarrow \frac{B'(x)}{y} = \frac{-2y \frac{B(x)}{y} + 3x}{y}$$

$$B' = \frac{-2B - 3x}{x} \Rightarrow B' + \frac{2B}{x} = -3 \quad (\text{linear ODE})$$

note: $G = \frac{1}{x} \nabla \phi$ $\phi = (2yx^2 + x^3)$

put B into z

$$z = \frac{-x + \frac{c}{x^2}}{y} \Rightarrow \boxed{zyx^2 + x^3 = c}$$



$$\mu = e^{\int \frac{2}{x} dx} = \mu = e^{\ln x^2} = x^2$$

$$\boxed{B = -x + \frac{c}{x^2}}$$

b) $(x-y)dx + zdy - xdz = 0$

a) existence: $G \cdot \text{curl } G = 0$, $G = (x-y)i + zj - xk$

$$\text{curl } G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & z & -x \end{vmatrix} = -1i - (-1)j + (-1)k = -i + j - k$$

$G \cdot \text{curl } G = -x + y + z + x = y + z \neq 0$ No soln !

ex for a given system, determine if a soln exist and find a soln (if exist)

$$\frac{\partial z}{\partial x} = 2yx^2$$

check $(z_x)_y = (z_y)_x$

$$\frac{\partial z}{\partial y} = e^{xy} + z$$

$$2z^2 + 2y \frac{\partial z}{\partial y} z_y = e^{xy} + z_x$$

$$2z^2 + 4yz \frac{\partial z}{\partial y} \neq e^{xy} + 2yz^2 \text{ . Not equal } \Rightarrow \text{no soln}$$

ex for a given system, determine if a soln exist and find a soln (if exist)

$$\begin{cases} \frac{dz}{dx} = yz \cos(xy) \\ \frac{dz}{dy} = xz \cos(xy) \end{cases}$$

check $(z_x)_y = (z_y)_x$

$$\begin{aligned} z \cdot \cos(xy) + y z_y \cos(xy) - y z_x \sin(xy) \\ = ? \\ z (\cos(xy) + x z_x \cos(xy)) - x z_y \sin(xy) \end{aligned}$$

$$y z_y \cos(xy) \stackrel{?}{=} x z_x \cos(xy)$$

$$\underbrace{y x z \cos(xy)}_{z_y} \cos(xy) \stackrel{?}{=} x \underbrace{(y z \cos(xy))}_{z_x} \cos(xy)$$

we see $z_{xy} = z_{yx}$ holds identically, soln exist.

we have $z_x = yz \cos(xy)$ $\frac{dz}{dx} = yz \cos(xy) \Rightarrow \int \frac{dz}{z} = \int y \cos(xy) dx$

$$\ln z = \frac{y}{x} \sin(xy) + B(y) \Rightarrow z = B(y) \sin(xy)$$

so $z_x = yz \cos(xy) \Rightarrow z = B(y) e^{\sin(xy)}$

$$B' e^{\sin(xy)} + B e^{\sin(xy)} x \cdot \cos(xy) = x B e^{\sin(xy)} \cos(xy)$$

$$\begin{aligned} B' e^{\sin(xy)} &= 0 & B &= C \\ z &= C \cdot e^{\sin(xy)} \end{aligned}$$

$$\int \cos xt \, dt = \frac{1}{x} \sin xt + C$$

2nd Order Quasilinear PDE

a second order quasilinear eqn is an eqn linear in second derivatives

$$a_{11}(x,y) u_{xx} + 2a_{12}(x,y) u_{xy} + a_{22}(x,y) u_{yy} + F(x,y, u, u_x, u_y) = 0$$

quadratic form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $a_{12} = a_{21}$

we consider classification of a 2nd order quasilinear PDE

idea we try to simplify the eqn by change of variables (how it works?)

we introduce new variables (ξ, η) : independent variables

$$\xi = \xi(x,y)$$

we rewrite the eqn in ξ and η - variables

$$\eta = \eta(x,y)$$

we have $u(\xi(x,y), \eta(x,y))$ we need expression for u_{xx}, u_{xy}, u_{yy}

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi (\xi_x(x,y), \eta(x,y)) \xi_x(x,y) + u_\eta (\xi(x,y), \eta(x,y)) \eta_x(x,y)$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = \dots$$

$$\begin{aligned} u_{xx} &= (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \cdot \xi_x + u_{\xi\xi} \xi_{xx} + (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x) \eta_x + u_{\eta\eta} \eta_{xx} \\ &= u_{\xi\xi} \cdot \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_{\xi\xi} \xi_{xx} + u_{\eta\eta} \eta_{xx} \end{aligned}$$

in the same way

$$u_{yy} = u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_{\xi\xi} \xi_{yy} + u_{\eta\eta} \eta_{yy}$$

$$u_{xy} = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \eta_x \xi_y) + u_{\eta\eta} \eta_x \eta_y + u_{\xi\xi} \xi_{xy} + u_{\eta\eta} \eta_{xy}$$

now subst into u_{xx}, u_{yy}, u_{xy}

$$a_{11} \mu_{xx} + 2a_{12} \mu_{xy} + a_{22} \mu_{yy} + \dots = 0$$

and collect coefficients of μ_{22} , μ_{12} and μ_{11}

$$\bar{a}_{11} \mu_{22} + 2\bar{a}_{12} \mu_{12} + a_{22} \mu_{11} + \dots = 0$$

$$\bar{a}_{11} = a_{11} \xi_x^2 + 2a_{12} \xi_x \xi_y + a_{22} \xi_y^2$$

$$\bar{a}_{12} = a_{11} \xi_x \eta_x + a_{12} (\xi_x \eta_y + \eta_x \xi_y) + a_{22} \xi_y \eta_y$$

$$\bar{a}_{22} = a_{11} \eta_x^2 + 2a_{12} \eta_x \eta_y + a_{22} \eta_y^2$$

we can make the eqn simpler by choosing ξ, η st some coefficients are zero

try to make $\bar{a}_{11} = 0$, we look for ξ st

$$a_{11} \xi_x^2 + 2a_{12} \xi_x \xi_y + a_{22} \xi_y^2 = 0 \quad (1)$$

claim $\xi(x,y)$ solve eqn (1) iff the equality $\xi(x,y) = C$ defines an implicit fn

satisfying $a_{11} dy^2 - 2a_{12} dx dy + a_{22} dx^2 = 0 \quad (2)$

pf assume $\xi(x,y) = C$ defines $y = y(x)$, then

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \quad \text{subst. } \frac{dy}{dx} \text{ into eqn (2)} \quad a_{11} \left(\frac{dy}{dx}\right)^2 - 2a_{12} \frac{dy}{dx} + a_{22} = 0$$

we get $a_{11} \left(-\frac{\xi_x}{\xi_y}\right)^2 - 2a_{12} \left(-\frac{\xi_x}{\xi_y}\right) + a_{22} = 0$ multiply by ξ_y^2 we get eqn (1)

$$a_{11} \xi_x^2 + 2a_{12} \xi_x \xi_y + a_{22} \xi_y^2 = 0 \quad \square$$

now let's solve eqn (2)

Solving (2)

$$a_{11} (dy)^2 - 2a_{12} dy dx + a_{22} (dx)^2 = 0$$

note
 $y' = f(x,y)$

first we solve $\frac{dy}{dx}$

$$a_{11} \left(\frac{dy}{dx} \right)^2 - 2a_{12} \frac{dy}{dx} + a_{22} = 0$$

$$\frac{dy}{dx} = \frac{a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \quad \text{assume } a_{11} \neq 0$$

13 Mart Pers

Recall classification of the 2nd order quasilinear PDE. We want to simplify a given eqn

$$a_{11}(x,y) u_{xx} + 2a_{12}(x,y) u_{xy} + a_{22}(x,y) u_{yy} + F(x,y,u,u_x,u_y) = 0$$

new variables, we have

$$\bar{a}_{11} u_{\xi\xi} + 2\bar{a}_{12} u_{\xi\eta} + \bar{a}_{22} u_{\eta\eta} + \bar{F}(\xi,\eta,u,u_\xi,u_\eta) = 0$$

$$\bar{a}_{11} = a_{11} \xi_x^2 + 2a_{12} \xi_x \xi_y + a_{22} \xi_y^2$$

$$\bar{a}_{12} = a_{11} \xi_x \eta_x + a_{12} (\xi_x \eta_y + \eta_x \xi_y) + a_{22} \xi_y \eta_y$$

$$\bar{a}_{22} = a_{11} \eta_x^2 + 2a_{12} \eta_x \eta_y + a_{22} \eta_y^2$$

try to make $\bar{a}_{11} = 0$

we proved

$$\frac{dy}{dx} = \frac{a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \quad \text{assume } a_{11} \neq 0$$

then we have 3 cases:

3 cases: for $\frac{dy}{dx} = \frac{a_{12} \mp \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$ assume $a_{11} \neq 0$

1) Hyperbolic eqn in Ω

$a_{12}^2 - a_{11}a_{22} > 0$ in Ω . In that case we have 2 solns

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \Rightarrow \xi(x, y)$$

hence we can make $\bar{a}_{11} = 0$ and $\bar{a}_{22} = 0$

so the canonical form of a hyperbolic eqn

$$\frac{dy}{dx} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \Rightarrow \eta(x, y)$$

$$\underline{\mu_{\xi\eta} = \tilde{F}(\xi, \eta, \mu, \mu_{\xi}, \mu_{\eta})}$$

2) Parabolic eqn in Ω

$a_{12}^2 - a_{11}a_{22} = 0$ in that case, we have only one soln

$$\frac{dy}{dx} = \frac{a_{12}}{a_{11}} \Rightarrow \xi(x, y) \text{ and for } \eta(x, y) \text{ we can take any smooth fn independent of } \xi(x, y)$$

Then $\bar{a}_{11} = 0$. It turns out that $\bar{a}_{12} = 0$

Indeed, $a_{12} = \sqrt{a_{11}a_{22}}$ now,

$$a_{11} \xi_x^2 + 2a_{12} \xi_x \xi_y + a_{22} \xi_y^2 = 0 \quad (\bar{a}_{11} = 0)$$

$$a_{11} \xi_x^2 + 2\sqrt{a_{11}a_{22}} \xi_x \xi_y + a_{22} \xi_y^2 = 0$$

$$(\sqrt{a_{11}} + \xi_x) + (\sqrt{a_{22}} + \xi_y)^2 = 0 \Rightarrow (\sqrt{a_{11}} \xi_x + \sqrt{a_{22}} \xi_y) = 0$$

$$\text{for } \bar{a}_{12} = a_{11} \xi_x \eta_x + a_{12} (\xi_x \eta_y + \xi_y \eta_x) + a_{22} \xi_y \eta_x$$

$$= a_{11} \xi_x \eta_x + \sqrt{a_{11}a_{22}} (\xi_x \eta_y + \xi_y \eta_x) + a_{22} \xi_y \eta_x$$

$$= \underbrace{(\sqrt{a_{11}} \xi_x + \sqrt{a_{22}} \xi_y)}_0 (\sqrt{a_{11}} \eta_x + \sqrt{a_{22}} \eta_y) \Rightarrow \bar{a}_{12} = 0$$

hence we have $\bar{a}_{22} = \mu_{\eta\eta} = F(\xi, \eta, \mu, \mu_{\xi}, \mu_{\eta}) \Rightarrow \underline{\mu_{\eta\eta} = \bar{F}(\xi, \eta, \mu, \mu_{\xi}, \mu_{\eta})}$
canonical form of parabolic eqn

3) Elliptic eqn in Ω

$a_{12}^2 - a_{11}a_{22} < 0$ in Ω In that case we have two complex solns

$$\frac{dy}{dx} = \frac{a_{12} + i\sqrt{a_{11}a_{22} - a_{12}^2}}{a_{11}} \Rightarrow \psi(x, y) = C$$

$$\frac{dy}{dx} = \frac{a_{12} - i\sqrt{a_{11}a_{22} - a_{12}^2}}{a_{11}}$$

hyperbolic ite +rs!

$$\psi(x, y) = C$$

two complex conjugate solns

$$\bar{\psi} = \overline{\psi}$$

$$\text{take } \xi = \frac{\psi + \bar{\psi}}{2} = \text{Re } \psi$$

$$\eta = \frac{\psi - \bar{\psi}}{2i} = \text{Im } \psi$$

$$a_{11}\psi_x^2 + 2a_{12}\psi_x\psi_y + a_{22}\psi_y^2 = 0$$

$$a_{11}(\xi_x + i\eta_x)^2 + 2a_{12}(\xi_x + i\eta_x)(\xi_y + i\eta_y) + a_{22}(\xi_y + i\eta_y)^2 = 0$$

$$(a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2) + 2i[a_{11}\xi_x\eta_x + 2a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y] - (a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2) = 0$$

$$\bar{a}_{11} + 2i\bar{a}_{12} - \bar{a}_{22} = 0 \quad \text{so} \quad \bar{a}_{11} = \bar{a}_{22} \quad \bar{a}_{12} = 0$$

so in ξ, η variables we get

$$\bar{a}_{11}\mu_{\xi\xi} + \bar{a}_{11}\mu_{\eta\eta} + \bar{F} = 0$$

$$\mu_{\xi\xi} + \mu_{\eta\eta} = \tilde{F} \quad \text{canonical form of an elliptic eqn}$$

Remark note form of the eqn depend on the domain Ω

ex consider the eqn $\frac{a_{11}}{y}\mu_{xx} - 2\mu_{xy} + x\mu_{yy} = 0$ $a_{12}^2 - a_{11}a_{22}$ (discriminant) Δ

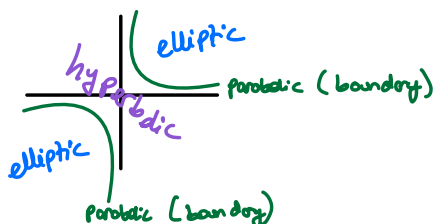
$$\Delta = (-1)^2 - yx = 1 - xy$$

hyperbolic $1 - xy > 0$

parabolic $1 - xy = 0$

elliptic $1 - xy < 0$

$xy > 1$
 $\frac{x}{y} > 1$ (elliptic)



$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = 0$$

$$\frac{dy}{dx} = \frac{2a_{12} \pm \sqrt{4a_{12}^2 - 4a_{11}a_{22}}}{2a_{11}} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

ex for a given eqn determine the type of eqn and transform it to canonical form

$$u_{xx} + 6u_{xy} - 16u_{yy} + u_x = 0 \quad (x, y) \in \mathbb{R}^2$$

$$a_{11} = 1 \quad a_{12} = 3 \quad a_{22} = -16 \quad \Delta = 9 - (1 \cdot (-16)) = 25 \quad (\text{hyperbolic eqn})$$

$$\xi, \eta ? \quad \text{we solve} \quad \frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

$$\xi: \frac{dy}{dx} = \frac{3+5}{1} = 8 \quad \int dy = \int 8 dx \Rightarrow y = 8x + c \quad y - 8x = c_1 \quad \boxed{\xi = y - 8x}$$

$$\eta: \frac{dy}{dx} = \frac{3-5}{1} = -2 \quad \int dy = \int -2 dx \Rightarrow y = -2x + c \quad y + 2x = c_2 \quad \boxed{\eta = y + 2x}$$

find the expressions for $u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ where $\eta = y + 2x$ $\xi = y - 8x$

$$u_x = \frac{\partial}{\partial x} u(\xi(x, y), \eta(x, y)) = u_\xi \xi_x + u_\eta \eta_x$$

$$u_x = u_\xi (-8) + u_\eta (2) = -8u_\xi + 2u_\eta = -8u_\xi(\xi(x, y), \eta(x, y)) + 2u_\eta(\xi(x, y), \eta(x, y))$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = 1 \cdot u_\xi + 1 \cdot u_\eta$$

$$u_{xx} = \frac{\partial}{\partial x} u_x = -8(u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) + 2(u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x)$$

$$u_{xx} = -8(u_{\xi\xi}(-8) + u_{\xi\eta}(2)) + 2(u_{\eta\xi}(-8) + u_{\eta\eta}(2))$$

$$= 64u_{\xi\xi} - 32u_{\xi\eta} + 4u_{\eta\eta}$$

$$u_{xy} = \frac{\partial}{\partial x} (u_y) = \frac{\partial}{\partial x} (u_\xi + u_\eta) = (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) + (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x)$$

$$= -8u_{\xi\xi} + 2u_{\xi\eta} - 8u_{\eta\xi} + 2u_{\eta\eta} =$$

$$u_{xy} = -8u_{\xi\xi} - 6u_{\xi\eta} + 2u_{\eta\eta}$$

$$u_{yy} = \frac{\partial}{\partial y} (u_y) = \frac{\partial}{\partial y} (u_\xi + u_\eta) = (u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y) + (u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y)$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

subst into original eqn

$$u_{xx} + 6u_{xy} - 16u_{yy} + u_x = 0$$

$$64u_{\xi\xi} - 32u_{\xi\eta} + 4u_{\eta\eta} + 6(-8u_{\xi\xi} - 6u_{\xi\eta} + 2u_{\eta\eta}) - 16(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 8u_\xi + 2u_\eta = 0$$

$$\boxed{100u_{\xi\eta} = \frac{8}{100}u_\xi - \frac{2}{100}u_\eta}$$

ex for a given eqn determine its type and transform into a canonical form

$$y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} - \frac{y^2}{x} u_x - \frac{x^2}{y} u_y = 0 \quad (x, y > 0)$$

$$a_{12}^2 - a_{11} a_{22} = (-xy)^2 - x^2 y^2 = 0 : \text{parabolic eqn}$$

$$a_{11} = y^2$$

$$a_{12} = -xy \quad \text{find } \xi, \eta$$

$$a_{22} = x^2$$

$$y^2 (dy)^2 + 2xy dy dx + x^2 (dx)^2 = 0$$

$$y^2 \left(\frac{dy}{dx}\right)^2 + 2xy \left(\frac{dy}{dx}\right) + x^2 = 0$$

$$\left(y \frac{dy}{dx} + x\right)^2 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y dy = \int -x dx \Rightarrow y^2/2 = -x^2/2 + c$$

$$\xi = x^2 + y^2, \eta = x$$

$$= x^2 + y^2 = c : \xi$$

$$\text{pick any } \eta, \text{ say } \eta = x$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = 2x u_\xi + u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = 2y u_\xi + u_\eta \cdot 0 = 2y u_\xi$$

$$u_{xx} = 2u_\xi + 2x(u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) + (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x)$$

$$= 2u_\xi + 2x(u_{\xi\xi} 2x + u_{\xi\eta}) + (u_{\xi\eta} 2x + u_{\eta\eta}) = 2u_\xi + 4x^2 u_{\xi\xi} + 4x u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(2y u_\xi) = 2y(u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x)$$

$$= 2y(2x u_{\xi\xi} + u_{\xi\eta}) = 4xy u_{\xi\xi} + 2y u_{\xi\eta}$$

$$u_{yy} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(2y u_\xi) = 2u_\xi + 2y(u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y)$$

$$= 2u_\xi + 2y(2y u_{\xi\xi} + 0) = 2u_\xi + 4y^2 u_{\xi\xi}$$

subst. into the eqn

$$y^2 (2u_\xi + 4x^2 u_{\xi\xi} + 4x u_{\xi\eta} + u_{\eta\eta}) - 2xy (4xy u_{\xi\xi} + 2y u_{\xi\eta})$$

$$+ x^2 (2u_\xi + 4y^2 u_{\xi\xi}) - \frac{y^2}{x} (2x u_\xi + u_\eta) - \frac{x^2}{y} (2y u_\xi) = 0$$

$$\Rightarrow (4x^2 y^2 - 8x^2 y^2 + 4x^2 y^2) u_{\xi\xi} + (4xy^2 - 4xy^2) u_{\xi\eta} + (y^2) u_{\eta\eta}$$

$$(2y^2 + 2x^2 - 2y^2 - 2x^2) u_\xi + (-2y^2 - 2x^2) u_\eta = 0 \Rightarrow y^2 u_{\eta\eta} - 2(x^2 + y^2) u_\eta = 0 \Rightarrow u_{\eta\eta} = \frac{2(x^2 + y^2)}{y^2} u_\eta$$

canonical form

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ex transform the eqn to the canonical form

$$u_{xx} - 2 \sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0 \quad x, y > 0$$

type $a_{12}^2 - a_{11} a_{22} = \sin^2 x + \cos^2 x = 1 > 0$, hyperbolic type

$$\begin{aligned} a_{11} &= 1 \\ a_{12} &= -\sin x \\ a_{22} &= -\cos^2 x \end{aligned}$$

solve $\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11} a_{22}}}{a_{11}} = \frac{-\sin x \pm \sqrt{1}}{1} = -\sin x + 1$ and $-\sin x - 1$

$$dy = (-\sin x + 1) dx = y = \cos x + x + c_1 \Rightarrow \xi = y - \cos x - x$$

$$dy = (-\sin x - 1) dx = \eta = \cos x - x + c_2 \Rightarrow \eta = y - \cos x + x$$

find $u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ while $\xi_x = \sin x - 1, \xi_y = 1$
 $\eta_x = \sin x + 1, \eta_y = 1$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi (\sin x - 1) + u_\eta (\sin x + 1)$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi \cdot 1 + u_\eta \cdot 1$$

$$u_{xx} = u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 $(\sin x - 1)^2$ $\sin x - 1$ $\sin x + 1$ $(\sin x + 1)^2$ $\cos x$ $\cos x$

$$u_{xx} = u_{\xi\xi} (\sin x - 1)^2 + 2u_{\xi\eta} (\sin^2 x - 1) + u_{\eta\eta} (\sin x + 1)^2 + u_\xi \cos x + u_\eta \cos x$$

$$u_{xy} = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

$$u_{xy} = u_{\xi\xi} (\sin x - 1) + u_{\xi\eta} (2 \sin x) + (\sin x + 1) u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

subst all terms in original eqn

$$(\sin x - 1)^2 u_{\xi\xi} - 2 \cos^2 x u_{\xi\eta} + (\sin x + 1)^2 u_{\eta\eta} + u_\xi \cos x + u_\eta \cos x$$

$$- 2 \sin x ((\sin x - 1) u_{\xi\xi} + 2 \sin x u_{\xi\eta} + (\sin x + 1) u_{\eta\eta})$$

$$- \cos^2 x (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - \cos x (u_\xi + u_\eta) = 0$$

we already know
 $0 \cdot u_{\xi\xi} + 0 \cdot u_{\eta\eta}$

$$[(\sin x - 1)^2 - 2 \sin x (\sin x - 1) - \cos^2 x] u_{\xi\xi} + (-2 \cos^2 x - 4 \sin^2 x - 2 \cos^2 x) u_{\xi\eta} + 0 \cdot u_{\eta\eta} = 0$$

$$-4u_{\xi\eta} = 0 \quad \square$$

Further simplifications

the equations with constant coefficients

In canonical form of hyperbolic case

$$a_3 \eta + a_1 \mu_\xi + a_2 \mu_\eta + c u = 0 \quad a_1, a_2, c \in \mathbb{R}$$

In canonical form of elliptic case

$$\mu_{\xi\xi} + \mu_{\eta\eta} + a_1 \mu_\xi + a_2 \mu_\eta + c u = 0 \quad a_1, a_2, c \in \mathbb{R}$$

In canonical form of parabolic case

$$\mu_{\xi\xi} + a_1 \mu_\xi + a_2 \mu_\eta + c u = 0 \quad a_1, a_2, c \in \mathbb{R}$$

in general we can remove two more terms μ_ξ, μ_η in hyperbolic and elliptic case, μ_ξ and μ_η in parabolic case
variables $V(\xi, \eta)$ by $u(\xi, \eta) = e^{p\xi + q\eta} V(\xi, \eta)$

we choose p, q st coeff of V_ξ and V_η are zero

ex simplify the eqn

$$u_{xx} - 2u_y - 2u_x - u = 0 \quad \text{setting } u = e^{px+qy} V(x, y)$$

$$u_x = p e^{px+qy} V + e^{px+qy} V_x$$

$$u_y = q e^{px+qy} V + e^{px+qy} V_y$$

$$u_{xx} = p^2 e^{px+qy} V + p e^{px+qy} V_x + p e^{px+qy} V_x + e^{px+qy} V_{xx}$$

now subst into the eqn

$$\cancel{e^{px+qy}} (p^2 V + 2pV_x + V_{xx}) - 2\cancel{e^{px+qy}} (qV + V_y) - 2\cancel{e^{px+qy}} (pV + V_x) - \cancel{e^{px+qy}} (V) = 0$$

$$V_{xx} + (2p-2)V_x - 2V_y + (p^2 - 2p - 2q - 1)V = 0$$

$$\left. \begin{array}{l} \text{choose } p \text{ st } 2p-2=0 \quad p=1 \\ \text{choose } q \text{ st } 1-2-2q-1=0 \quad q=-1 \end{array} \right\} p=1 \quad q=-1 \Rightarrow \text{if } u = e^{x-y} V \text{ or } v = e^{y-x} u$$

then V satisfies

$$V_{xx} - 2V_y = 0$$

ex simplify the eqn

$$u_{xx} + u_{yy} + 4u_x - 2u_y + u = 0 \quad \text{by using } u = e^{px+qy} \cdot v$$

$$u_x = p e^{px+qy} v + e^{px+qy} v_x$$

$$u_y = q e^{px+qy} v + e^{px+qy} v_y$$

$$u_{xx} = p^2 e^{px+qy} v + p e^{px+qy} v_x + p e^{px+qy} v_x + e^{px+qy} v_{xx}$$

$$u_{yy} = q^2 e^{px+qy} v + 2q e^{px+qy} v_y + e^{px+qy} v_{yy}$$

now subst into original eqn

$$\cancel{e^{px+qy}} (v_{xx} + 2p v_x + p^2 v) + \cancel{e^{px+qy}} (v_{yy} + 2q v_y + q^2 v) + 4\cancel{e^{px+qy}} (p v + v_x) - 2\cancel{e^{px+qy}} (q v + v_y) + \cancel{e^{px+qy}} v = 0$$

$$v_{xx} + v_{yy} + (2p+4)v_x + (2q-2)v_y + (p^2+q^2+4p-2q+1)v = 0$$

$$4+1-8-2+1=0$$

$$\text{set } 2p+4=0 \Rightarrow p=-2$$

$$\text{set } 2q-2=0 \Rightarrow q=1$$

$$v_{xx} + v_{yy} - 4v = 0$$

if $u = e^{-2x+y} v$ then v satisfies the following eqn

$$v_{xx} + v_{yy} - 4v = 0$$

WAVE EQUATION

$$u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < +\infty \quad t > 0 \quad c > 0$$

note that this eqn is not canonical for

$$a_{11}=1 \quad a_{12}=0 \quad a_{22}=-c^2 \Rightarrow a_{12}^2 - a_{11}a_{22} = c^2 > 0 \quad \text{hyperbolic eqn}$$

transform it to canonical form

$$\frac{dx}{dt} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \frac{dt}{dx} = \pm c \Rightarrow dx = c dt \quad dx = -c dt$$

$$x = ct + c_1$$

$$x = -ct + c_2$$

$$\xi = x - ct$$

$$\eta = x + ct$$

use ξ, η and find u_x, u_y, u_{xx}, u_{yy}

$$u_x = u_\xi \xi_x + u_\eta \eta_x = c u_\xi + c u_\eta$$

$$u_t = u_\xi \xi_t + u_\eta \eta_t = u_\xi + u_\eta$$

$$u_{xx} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$$

$$u_{tt} = \dots \quad \text{then we get } u_{\xi\eta} = 0$$