# Non Homogeneous System

$$x' = Ax + g(t)$$

$$x_1' = -5x_1 + x_2 + 6e^{2t}$$
$$x_2' = 4x_1 - 2x_2 - e^{2t}$$
$$\Rightarrow x' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} x + \begin{bmatrix} 6e^{2t} \\ -e^{2t} \end{bmatrix}$$

$x_{general} = x_{homogen} + x_{particular}$

$x_p \to$ Undet Coeff

$x_p \to$ Variation of Parameter Powerful ($\int$)

$\underline{x_h}$ $|A - \lambda I| = 0$

$$\begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda+6)(\lambda+1) = 0 \quad \lambda = -1, -6$$

$\underline{\lambda = -6}$

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ \cancel{4x_1 + 4x_2 = 0} \end{matrix} \qquad \begin{matrix} x_1 = -x_2 \\ -a \quad\ \ a \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \cancel{-4x_1 + x_2 = 0} \\ 4x_1 - x_2 = 0 \end{matrix} \qquad \begin{matrix} 4x_1 = x_2 \\ \frac{a}{4} \quad\ \ a \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$

$$\left. x_h = c_1 e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1/4 \\ 1 \end{pmatrix} \right\} y \text{ homogeneous}$$

now find $y_p$ by VOP

$\underline{y_p}$ $x_{Homogen} \to$ fundamental matrix $= \Psi(t)$

2x2 matrislerin tersi

$$\begin{matrix} a & b \\ c & d \end{matrix} \to \begin{matrix} d & -b \\ -c & a \end{matrix}$$

$$\Psi(t) = \begin{bmatrix} -e^{-6t} & \frac{e^{-t}}{4} \\ e^{-6t} & e^{-t} \end{bmatrix}$$ now find $\Psi^{-1}$

$$\Psi^{-1} = \frac{\begin{pmatrix} e^{-t} & \frac{-e^{-t}}{4} \\ -e^{-6t} & -e^{-6t} \end{pmatrix}}{\det(\Psi(t))} = \frac{\begin{pmatrix} e^{-t} & \frac{-e^{-t}}{4} \\ -e^{-6t} & -e^{-6t} \end{pmatrix}}{-e^{-7t} - \frac{e^{-7t}}{4}} = \begin{pmatrix} \frac{-4}{5}e^{6t} & \frac{1}{5}e^{6t} \\ \frac{4}{5}e^{t} & \frac{4}{5}e^{t} \end{pmatrix}$$

Multiply by $g(t) = \begin{pmatrix} 6e^{2t} \\ -e^{2t} \end{pmatrix}$

$$\Psi_p(t) = \Psi(t) \cdot \int \Psi^{-1}(t)\, g(t)\, dt$$

$$\Psi^{-1} \cdot g(t) = \begin{pmatrix} \frac{-4}{5}e^{6t} & \frac{1}{5}e^{6t} \\ \frac{4}{5}e^{t} & \frac{4}{5}e^{t} \end{pmatrix}\begin{pmatrix} 6e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} \frac{-24}{5}e^{8t} - \frac{1}{5}e^{8t} \\ \frac{24}{5}e^{3t} - \frac{4}{5}e^{3t} \end{pmatrix} = \begin{pmatrix} -5e^{8t} \\ 4e^{3t} \end{pmatrix}$$

now integrate this

$$\underbrace{\int \begin{pmatrix} -5e^{8t} \\ 4e^{3t} \end{pmatrix} dt}_{} = \underbrace{\begin{pmatrix} -5/8\, e^{8t} \\ 4/3\, e^{3t} \end{pmatrix}}_{\int \Psi^{-1}(t)\, g\, dt} \Rightarrow \Psi_p = \begin{bmatrix} -e^{-6t} & \frac{e^{-t}}{4} \\ e^{-6t} & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} -5/8\, e^{8t} \\ 4/3\, e^{3t} \end{bmatrix} = \underbrace{e^{2t}\begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix}}_{y \text{ particular}}$$

# Non Homogeneous System by Undetermined Coeff

$$X' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} X + \begin{pmatrix} 2t \\ -4t \end{pmatrix}$$

$\underline{X_h}$ $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = 0$ $\lambda = 4$ $\lambda = -1$

$\underline{\lambda = 4}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$ $\underline{\lambda = -1}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$X_h = c_1 e^{4t} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

now find $y_p$ by UCM $X' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} X + \begin{pmatrix} 2t \\ -4t \end{pmatrix}$ guess method

guess $X_p = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix}$ ama $\begin{pmatrix} t \\ t^2+1 \end{pmatrix}$ olsaydı ikisi de 2. dereceden yazmalısın !

$\begin{pmatrix} t \\ e^{3t} \end{pmatrix} \rightarrow \begin{pmatrix} At + B + C.e^{3t} \\ Dt + E + Fe^{3t} \end{pmatrix}$ demek zorundasın !

$X_p = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix}$ put it into original ODE

$$X_p' = AX \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} At + B \\ Ct + D \end{pmatrix} + \begin{pmatrix} 2t \\ -4t \end{pmatrix}$$

$$= \begin{pmatrix} At + B + 2Ct + 2D \\ 3At + 3B + 2Ct + 2D \end{pmatrix} + \begin{pmatrix} 2t \\ -4t \end{pmatrix}$$

① $A = At + B + 2Ct + 2D + 2t$
② $C = 3At + 3B + 2Ct + 2D - 4t$

$$\Rightarrow \left.\begin{matrix} A + 2C = -2 \\ B + 2D = A \end{matrix}\right\} ① \qquad \left.\begin{matrix} 3A + 2C - 4 = 0 \\ 3B + 2D = C \end{matrix}\right\} ②$$

Çöz $X_p$ bulmuş olacaksın

$A = 3$ , $B = \frac{-11}{4}$ , $C = \frac{-5}{2}$ , $D = \frac{23}{8}$

$$X_{gen} = c_1 e^{4t} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3t - \frac{11}{4} \\ \frac{-5t}{2} + \frac{23}{8} \end{pmatrix}$$

$$X_p = \begin{pmatrix} 3t - 11/4 \\ -5/2t + 23/8 \end{pmatrix}$$

# 3x3 MATRIX $X' = AX$ ALL TYPES

① $\lambda_1 \neq \lambda_2 \neq \lambda_3 \in \mathbb{R}$ : all distinct real 3 eigenvalues

$$X' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} X \qquad \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0 \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix}$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{matrix} x_2 = 0 \\ x_2 + x_3 = 0 \\ 2x_3 = 0 \end{matrix} \Rightarrow x_3 = 0 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{matrix} -x_1 + x_2 = 0 \\ x_3 = 0 \\ x_3 = 0 \end{matrix} \qquad \begin{matrix} x_1 = x_2 \\ x_3 = 0 \end{matrix} > v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 3}$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{matrix} -2x_1 + x_2 = 0 \\ -x_2 + x_3 = 0 \end{matrix} \qquad \begin{matrix} x_2 = 2x_1 \\ x_2 = x_3 \end{matrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

fundamental matrix $\Psi(t) = \begin{pmatrix} e^{t} & e^{2t} & e^{3t} \\ e^{t} & e^{2t} & 2e^{3t} \\ 0 & 0 & 2e^{3t} \end{pmatrix}$

2a Repeated but still good : algebraic multiplicity (2+1) $\operatorname{diag}(\lambda, \lambda, \mu)$

$$X' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} X$$

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda)(-1-\lambda) = 0$$

$\lambda_{1,2} = 2 \quad AM = 2$

$\lambda_3 = -1 \quad AM = 1$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{matrix} x_3 = 0 \\ x_3 = 0 \\ -3x_3 = 0 \end{matrix} \Rightarrow \underline{x_1, x_2 \text{ free}} \Rightarrow \text{null space 2dim} \Rightarrow \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{2 eigenvectors } v_1 \ v_2}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{matrix} 3x_1 + x_3 = 0 \\ 3x_2 + x_3 = 0 \end{matrix} \Rightarrow \underset{1}{3x_1} = \underset{1}{3x_2} = \underset{-3}{-x_3} \qquad \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = v_3$$

$$X(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

The reason we didn't write $t \cdot e^{2t}$ is because $\lambda = 2$ already comes with 2 linearly independent vectors. The $te^{2t}$ only needed when eigenvalues $GM < AM$

## (2b) Repeated Double AM ⇒ 2+1 , double eigenvalue GM = 1 (one 2x2 Jordan block)

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$X' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \ , \quad \lambda_3 = -1$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_2 = 0$
$-3x_3 = 0 \Rightarrow x_3 = 0$
pick $x_1 = 1$
$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

only $x_1$ free we dont get two lin indep eigenvectors here
we need generalized eigenvector

generalized eigenvector

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} \Rightarrow$$

$x_2 = 1$
$-3x_3 = 0$
pick $x_1 = 0$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

for simplicity, you can say $x_1 = 1$ too, just put in eqn and don't forget to check lin. independent

$\underline{\lambda = -1}$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$$

$3x_1 + x_2 = 0 \Rightarrow \underset{0}{3x_1} = \underset{0}{-x_2}$
$3x_2 = 0$
$x_3$ free, take $x_3 = 1$

$\Bigg\} \ v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

fundamental soln set

$$x_1(t) = v_1 e^{2t} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2(t) = t v_1 e^{2t} + v_2 e^{2t} = t e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}$$

$$x_3(t) = e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

fundamental matrix

$$\begin{pmatrix} e^{2t} & t e^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

$$X_{gen} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \left[ t e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$X_{gen} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

3a Repeated Triple , $AM = 3 = GM$ , $\underline{\lambda_1 = \lambda_2 = \lambda_3}$ $(\lambda I_3)$

$$x' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} x$$

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 4$$

A is scalar matrix $AM = 3 = GM$

$\underline{\lambda = 4}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= e^{4t} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{4t} c$$

## (3b) Repeated Triple $\lambda_1=\lambda_2=\lambda_3$ $\underline{\text{BUT} \quad GM=2<3=AM}$ (extra $te^{\lambda t}$)

$$x' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)^3 = 0 \qquad \lambda_1=\lambda_2=\lambda_3=2$$

$$AM(\lambda=2) = 3$$

$\underline{\lambda=2}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \left.\begin{array}{l} x_2 = 0 \\ x_1, x_3 \text{ free} \end{array}\right\} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underbrace{GM(\lambda=2)}_{2} < \underbrace{AM(\lambda=2)}_{3}$$

One eigenvector missing

generalised eigenvector:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_2} \quad \cancel{\Rightarrow} \quad \begin{array}{l} x_2 = 0 \\ 0 \neq 1 \end{array}$$

it did not work ☹

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} \quad \checkmark \Rightarrow \quad x_2 = 1$$

choose the simplest $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$x_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_2(t) = e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad x_3(t) = e^{2t}\left(t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

$$x_{gen} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

## 3c Repeated Triple $AM = 3$, $GM = 1$ (Max defect $te^{\lambda t}$, $\frac{t^2}{2}e^{\lambda t}$)

$$x' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} x$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^3 = 0 \qquad \lambda_1 = \lambda_2 = \lambda_3 = 2$$

$AM = 3$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix}, \; x_1 \text{ free} \quad \overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

only one eigenvector

two eigenvector missing

$GM = 1 < 3 = AM$

first generalized eigenvector

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 = 1 \\ x_3 = 0 \end{matrix} \quad x_1 \text{ free}, \text{ take } x_1 = 0$$

hence $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Second generalized eigenvector

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix} \quad x_1 \text{ free}, \text{ take } x_1 = 0$$

hence $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

now check the chain

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{(A-2I)} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} = 0 \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{(A-2I)} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_2} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{(A-2I)} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3} - \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2}$$

$$x_1(t) = e^{2t}\overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \qquad x_2(t) = e^{2t}\left(\overset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + t\overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}\right) = e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

$$x_3(t) = e^{2t}\left(\underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} + t\underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \frac{t^2}{2}\underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}\right)$$

$$x_{gen} = c_1 x_1 + c_2 x_2 + c_3 x_3$$

# 2 complex - 1 Real Eigenvalue

$$X' = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix} X$$

eigenvalues $\lambda_1 = -3$ , $\lambda_{2,3} = 1 \mp 2i$
$\alpha + i\beta$

$\underline{\lambda = -3}$

eigenvector $\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 1 \mp 2i$
$1+2i$
$\alpha = 1$ , $\beta = 2$

$$\begin{pmatrix} -3-1-2i & 0 & 0 \\ 0 & 1-1-2i & -2 \\ 0 & 2 & 1-1-2i \end{pmatrix} = \begin{pmatrix} -4-2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$(-4-2i)\, x_1 = 0$ $\quad x_1 = 0$ $\quad 2x_2 + (-2i)\, x_3 = 0$

$(-2i)\, x_2 - 2x_3 = 0$ $\quad (-2i)\underset{1}{x_2} = 2\underset{-i}{x_3}$

$v = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{b} i$

$$x_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2(t) = e^{t} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \sin 2t \right)$$

$$x_3(t) = e^{t} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cos 2t \right)$$

$\alpha + i\beta$ , $a, b$
$\alpha \to 1$, $\beta \to 2$ ; $a \to$ real, $b \to$ imaginary

$$X_{gen} = c_1\, e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\, e^{t} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \sin 2t \right) + c_3\, e^{t} \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cos 2t \right)$$

# $e^{At}$ matrisi bulma

1) find eigenvalues / eigenvectors
2) write $A = PDP^{-1}$
3) find $P^{-1}$
4) $e^{At} = P e^{Dt} P^{-1}$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \qquad e^{At} \ ?$$

eigenvalues $\lambda_1 = -2$ , $\lambda_2 = -1$

eigenvectors $\underbrace{\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}}_{v_1}$ $\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{v_2}$

sıra önemli

1st eigenvalue

$$P = \begin{array}{cc} v_1 & v_2 \\ \end{array} \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

2nd eigenvalue

$A = PDP^{-1}$

$$A = \underbrace{\begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}_{P^{-1}}$$

$$\begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix} = \frac{1}{\frac{-1}{2}+1} \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}}_{P^{-1}}$$

$$e^{At} = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

$$\begin{bmatrix} -\frac{e^{-2t}}{2} & -e^{-t} \\ e^{-2t} & e^{-t} \end{bmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix} = \underbrace{\begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}}_{e^{At}}$$

# $e^{At}$ matrisi bulma

$A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$ find $e^{At}$ and solve IVP $x' = Ax$ with $x(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda_1 = 1 \rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 7 \rightarrow v_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

KOLAY

so the general soln $x = c_1 e^{t} v_1 + c_2 e^{7t} v_2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} e^{t} & 5e^{7t} \\ -e^{t} & e^{7t} \end{pmatrix}}_{\Psi(t)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\Psi(0) = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}$ , $\Psi^{-1}(0) = \frac{1}{6} \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$

$$e^{At} = \Psi(t)\Psi^{-1}(0) = \underbrace{\frac{1}{6} \begin{pmatrix} e^{t} + 5e^{7t} & -5e^{t} + 5e^{7t} \\ -e^{t} + e^{7t} & 5e^{t} + e^{7t} \end{pmatrix}}_{e^{At}}$$

$x(t) = e^{At} x(0)$

Çözüm bulmak için initial value ile çarp

$$\frac{1}{6} \begin{pmatrix} e^{t} + 5e^{7t} & -5e^{t} + 5e^{7t} \\ -e^{t} + e^{7t} & 5e^{t} + e^{7t} \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -e^{t} + 25e^{7t} \\ e^{t} + 5e^{7t} \end{pmatrix} =$$

$$= \underline{\frac{e^{t}}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{e^{7t}}{6} \begin{pmatrix} 25 \\ 5 \end{pmatrix}}$$

**$e^{At}$ is a fundamental matrix for the system x'=Ax such that the solution of the inital value problem x'=Ax, x(0)=$x_0$ is $e^{At}x_0$**

Exercise: Prove this statement using Theorem 8 and 9 in the book.

$e^{At}$ is a fundamental matrix for $x' = Ax$

let $\varphi(t)$ be any other fund. matrix

thus we have 2 fund. matrices $e^{At} = \varphi(t)$

Therefore $\exists$ constant $(\neq 0)$ matrix $C$ st $e^{At} = \varphi(t) . C$

evaluate at $t=0$ : $e^{A.0} = I = \varphi(0) c \Rightarrow c = \varphi(0)^{-1}$

Hence $e^{At} = \varphi(t) \varphi(0)^{-1} \quad \forall t$

the general soln in old fund. matrix

$$x(t) = \varphi(t) c \qquad c \in \mathbb{R}^n$$

$$x(t) = \varphi(t) c = e^{At} (\underbrace{\varphi(0) c}_{x_0})$$

hence $x(t) = e^{At} x_0$

verify $t=0 \Rightarrow x(0) = e^{A.0} x_0 = I x_0 = x_0$ so IVP satisfies

the unique soln of $x' = Ax$ , $x(0) = x_0$ $\Rightarrow x(t) = e^{At} x_0$

$$x' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x \qquad e^{At} \text{ find}$$

eigenvalues $\lambda^2 + 1 = 0 \qquad \lambda = \mp i$

eigenvectors $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad (-i)\underset{1}{x_1} + \overset{i}{x_2} = 0$

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} i \qquad \text{hence} \quad x_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\psi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad x(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow x(0)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = \psi(t)\,\psi(0)^{-1} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} x \qquad \text{find } e^{At}$$

2 complex 1 Real

Real $\lambda = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

complex $\lambda = \mp i \qquad v = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{b} i$

$\to$ solns $\underbrace{\begin{pmatrix} -\cos t \\ -\sin t \\ \cos t \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} -\sin t \\ \cos t \\ \sin t \end{pmatrix}}_{v_3} \Rightarrow \begin{pmatrix} e^t & -\cos t & -\sin t \\ e^t & -\sin t & \cos t \\ e^t & \cos t & \sin t \end{pmatrix}$

$$x(0) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow x^{-1}(0) = \begin{pmatrix} 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}$$

$$e^{At} = x(t)\cdot x^{-1}(0) = \begin{pmatrix} \frac{e^t + \sin t + \cos t}{2} & -\sin t & \frac{e^t - \cos t + \sin t}{2} \\ \frac{e^t + \sin t - \cos t}{2} & \cos t & \frac{e^t - \sin t - \cos t}{2} \\ \frac{e^t - \sin t - \cos t}{2} & \sin t & \frac{e^t - \sin t + \cos t}{2} \end{pmatrix}$$

$$x' = \begin{pmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{solve by } e^{At}$$

the shift $-2$ $\Rightarrow$ $A + 2I = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{N}$ and $N^2 = 0$

nilpoint of index $= 2$

$$e^{At} = e^{(-2I + N)t}$$

$$= e^{-2tI} \cdot e^{Nt} = e^{-2t} \cdot e^{Nt}$$

$$e^{Nt} = I + tN + \frac{t^2}{2} N^2 = I + tN + \begin{pmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{pmatrix}$$

$$e^{At} = e^{-2t} \begin{pmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{pmatrix}$$

IVP: $$e^{-2t} \begin{pmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ 4t+1 \\ t-1 \end{pmatrix}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$