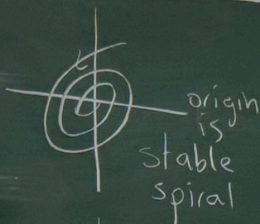
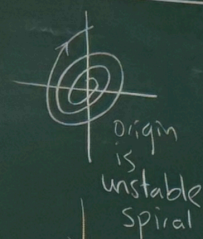


Complex eigenvalues $\lambda = \alpha + i\beta$

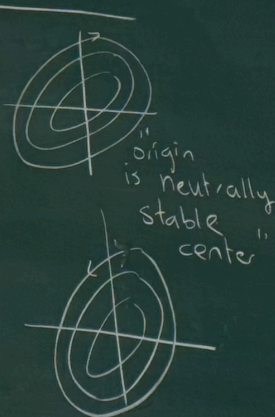
$\alpha = \text{Re}(\lambda) < 0$



$\alpha = \text{Re}(\lambda) > 0$



$\alpha = \text{Re}(\lambda) = 0$



To decide about direction of arrow: Pick a suitable point (e.g. $\vec{e}_1 = (1,0)$ or $\vec{e}_2 = (0,1)$ or sth else) and substitute it in $\vec{x}' = A\vec{x}$ and draw the vector at chosen point with slope $\vec{x}'$. The direction of the vector will give us direction of arrow.

ex $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$ solve, draw phase portrait

$$\det(A - \lambda I) = (1-\lambda)(-3-\lambda) + 5 = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\text{so, } \lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\text{let } w_1 = 1 \quad w_2 = 2-i$$

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\lambda_1 = -1+i \rightarrow (A - (-1+i)I) \vec{v} = \vec{0}$$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \rightarrow \begin{pmatrix} 2-i & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} w_1 \text{ leading} \\ w_2 \text{ free} \end{matrix}$$

$$\begin{aligned} \vec{x}_1(t) &= e^{(-1+i)t} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\ &= e^{-t} (\cos t + i \sin t) \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) \end{aligned}$$

$$\Rightarrow \underbrace{e^{-t} (\cos t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix})}_{\text{Re}(\vec{x}_1) \text{ is a real soln.}} + i \underbrace{e^{-t} (\cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ 2 \end{pmatrix})}_{\text{Im}(\vec{x}_1) \rightarrow \text{is a real soln.}}$$

$$\Rightarrow x_g = c_1 \text{Re}(\vec{x}_1) + c_2 \text{Im}(\vec{x}_1) \quad [\text{real solns}]$$

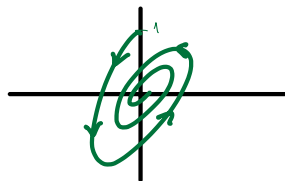
$$\Rightarrow x_g = c_1 e^{-t} (\cos t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2 e^{-t} (\cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ 2 \end{pmatrix})$$

Phase portrait

$$\vec{x} = \begin{pmatrix} 1 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

lets change

??



$$\vec{x}_1, \vec{x}_2 = \overline{\vec{x}_1}$$

$$\vec{z} = \vec{a} + i\vec{b}$$

ϕ_1, ϕ_2

$$\begin{aligned} \vec{z} &= \vec{a} + i\vec{b} \\ \vec{z} &= \vec{a} - i\vec{b} \\ \frac{\phi_1 + \bar{\phi}_1}{2} &= \frac{z + \bar{z}}{2} = \vec{a} = \text{Re}(\vec{z}) \\ \frac{\phi_1 - \bar{\phi}_1}{2i} &= \frac{z - \bar{z}}{2i} = \vec{b} = \text{Im}(\vec{z}) \end{aligned}$$

$c_1 = \frac{1}{2i}$

$c_2 = -\frac{1}{2i}$

$$\phi = \phi(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$2x + 5y = 4$$

$$2x = 8$$

$$-2x - 2y = 6$$

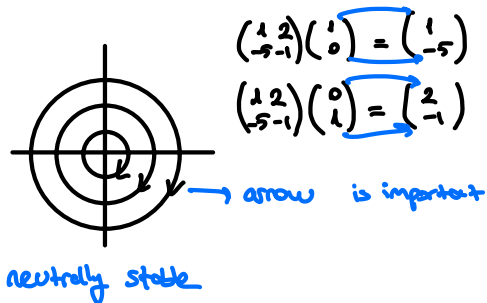
$$\begin{aligned} 7y &= -2 \\ y &= -2/7 \end{aligned}$$

ex $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \vec{x}$

$$\det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) + 10 = 0$$

$$= \lambda^2 + 9 = 0 \quad \lambda = \pm 3i \quad \text{"origin center"}$$

$$\alpha = \operatorname{Re}(\lambda) = 0$$



$\lambda = 3i$

$$(A - 3iI) \vec{w} = \vec{0}$$

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \xrightarrow{\frac{1+3i}{2} R_1 + R_2} \begin{pmatrix} 1-3i & 2 \\ 0 & 0 \end{pmatrix}$$

leading w_1
 w_2 free
 if not 0,0 then you made mistake

$$w_1 = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$\vec{x}_1(t) = e^{3it} \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right)$
 or $\parallel$
 $(\cos 3t + i \sin 3t)$

$$\Rightarrow \underbrace{\left(\cos(3t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin(3t) \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right)}_{\operatorname{Re}(\vec{x}_1)} + i \underbrace{\left(\cos(3t) \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \sin(3t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)}_{\operatorname{Im}(\vec{x}_1)}$$

$\vec{x}_g = c_1 \operatorname{Re}(\vec{x}_1) + c_2 \operatorname{Im}(\vec{x}_1)$
 $\downarrow$
 real

exercise $\vec{x}' = \begin{pmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$

solve the system, write general solns $(\lambda_1 = 2, \lambda_{2,3} = \pm i)$
 pick this pick one only

case III Repeated Eigenvectors

$$\operatorname{alg}(\lambda) > 1$$

$$\vec{x}' = A \vec{x}$$

$$\det(A - \lambda I) = 0$$

$$\operatorname{alg}(\lambda) = k > 1$$

two possibilities:

① k linearly independent eigenvectors

$$\operatorname{alg}(\lambda) = k = m(\lambda)$$

of free variable = # of lin. indep. eigenvectors

two solutions will be

$$\vec{x}_1(t) = e^{\lambda t} \vec{v}_1$$

$$\vec{x}_2(t) = e^{\lambda t} \vec{v}_2$$

$$\vec{x}_k(t) = e^{\lambda t} \vec{v}_k$$

② there are less than k linearly independent eigenvectors :<

$(A - \lambda I)\vec{v} = \vec{0}$ gives us eigenvectors less than k

$$m(\lambda) < \text{alg}(\lambda) = k$$

Question: how can we obtain the missing eigenvectors (generalized eigenvectors) [Jordan form]

(if $m(\lambda) < \text{alg}(\lambda)$ then it's not diagonalizable right now.)

Defn: to find eigenvectors we solve $(A - \lambda I)\vec{v} = \vec{0}$, actually we are

looking at **kernel of $(A - \lambda I)$**

$$\text{Ker}(A - \lambda I) = \{ \vec{v} \mid (A - \lambda I)\vec{v} = \vec{0} \} = V_\lambda \text{ (kernel notation)}$$

= { set of lin. ind. eigenvectors
corresponding to λ of A }

$$\dim(V_\lambda) = m(\lambda) = \# \text{ free variables in } (A - \lambda I)\vec{v} = \vec{0}$$

lin. ind. eigenvectors.

ex $\text{alg}(\lambda) = m(\lambda)$

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

$$\det(A - \lambda I) = -(\lambda + 1)^2(\lambda - 2) = 0$$

$\lambda = -1 \quad \lambda = 2 \quad (\text{single})$

$$\text{alg}(\lambda = -1) = 2$$

$$\text{alg}(\lambda = 2) = 1 = m(\lambda = 2) \text{ (repeated)}$$

$$\lambda = 2 \rightarrow (A - 2I)\vec{v} = \vec{0} \quad \text{find } \text{Ker}(A - 2I)$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$v_1 \quad v_2 \quad \text{lead}$
 $v_3 \quad \text{free}$

$$v_1 - \frac{1}{2}v_2 + \frac{1}{2}v_3 = 0 \rightarrow v_1 = 0$$

$$v_2 - v_3 = 0 \Rightarrow v_2 = v_3$$

$$\vec{v} = \begin{pmatrix} 0 \\ v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Ker}(A - 2I) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$V_{\lambda=2}$

$$\dim(V_{\lambda=2}) = m(\lambda=2) = 1$$

$$\vec{x} = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \quad (A+I)\vec{\mu} = \vec{0} \quad \text{find } \ker(A+I)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} v_1 \text{ lead} \\ v_2, v_3 \text{ free} \end{array}$$

$$\begin{aligned} \mu(\lambda = -1) &= 2 = \dim(V_{\lambda=-1}) \\ &\parallel \\ \text{alg}(\lambda = -1) &\quad \text{☺} \end{aligned}$$

$$v_1 + v_2 + v_3 = 0 \Rightarrow v_1 = -v_2 - v_3$$

$$\vec{\mu} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$V = \ker(A+I) = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \dim(\ker(A+I)) = 2 = \mu(\lambda = -1)$$

$$\vec{x}_2(t) = c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_3(t) = c_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x}_0 = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$$

here observe A is diagonalizable $\Rightarrow \exists$ invertible matrix P st $D = P^{-1}AP$

$$\text{where } D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Question what happens if A is not diagonalizable? or what happens

if $\mu(\lambda) < \text{alg}(\lambda)$, that is if there is not enough # of linearly

independent eigenvectors?

in this case find jordan form $P^{-1}AP = J$

Thm $\forall A_{n \times n}$, $\exists$ invertible matrix P st $P^{-1}AP = J$

Jordan form

$$\left[\begin{array}{ccc|ccc|ccc} \lambda_1 & 1 & 0 & & & & & & \\ & \lambda_1 & 1 & & & & & & \\ 0 & & & \lambda_2 & 1 & 0 & & & \\ & & & & \lambda_2 & 1 & & & \\ & & & 0 & & & \lambda_3 & 1 & 0 \\ & & & & & & & \lambda_3 & 1 \\ & & & & & & 0 & & \lambda_n \\ & & & & & & & & \lambda_n \end{array} \right]$$

$$J = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & \dots & 0 \\ & \lambda_i & 1 & \\ & & \ddots & \lambda_i \end{pmatrix}$$

ex

$$A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix}$$

repeated

$$\lambda = 1, 2, 4, 4$$

$\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4$

$$\text{alg}(1) = 1 = \mu(1) \rightarrow J_{\lambda=1} = [1]_{1 \times 1}$$

$$\text{alg}(2) = 1 = \mu(2) \rightarrow J_{\lambda=2} = [2]_{1 \times 1}$$

$$\text{alg}(4) = 2 \rightarrow \mu(\lambda=4) = 2 \text{ (smiley face)} \rightarrow J_{2 \times 2} = \left(\begin{array}{c|c} 4 & 0 \\ \hline 0 & 4 \end{array} \right)_{2 \times 2}$$

Remark:

1) $\text{alg}(\lambda)$ = size of jordan form

2) $\mu(\lambda)$ = # jordan blocks in jordan form

$$A \rightarrow J = \begin{pmatrix} J_{\lambda=1} & & \\ & J_{\lambda=2} & \\ & & J_{\lambda=4} \end{pmatrix} = \begin{pmatrix} \boxed{1} & & & 0 \\ & \boxed{2} & & \\ & & \boxed{4 \ 0} & \\ 0 & & \boxed{0 \ 4} & \end{pmatrix} \text{ or } \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & \boxed{4 \ 1} & \\ 0 & 0 & 0 & \boxed{4} \end{pmatrix}$$

ex $\det(A - \lambda I) = \lambda(\lambda-1)^2(\lambda-2) = 0$

$$\lambda=0 \quad \text{alg}(\lambda=0) = 1 = \mu(\lambda=0) \rightarrow J_{1 \times 1} = [0]$$

$$\lambda=2 \quad \text{alg}(\lambda=2) = 1 = \mu(\lambda=2) \rightarrow J_{1 \times 1} = [2]$$

$$\lambda=1 \quad \text{alg}(\lambda=1) = 2 \begin{cases} \text{smiley face} & \mu(\lambda=1)=2 \rightarrow J_{2 \times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \text{frowny face} & \mu(\lambda=1)=1 \rightarrow J_{2 \times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{cases}$$

$$J = \begin{pmatrix} \boxed{0} & & & \\ & \boxed{2} & & \\ & & \boxed{1 \ 1} & \\ & & 0 & 1 \end{pmatrix} \quad \text{frowny face} \quad \text{or} \quad J = \begin{pmatrix} \boxed{0} & & & \\ & \boxed{2} & & \\ & & \boxed{1} & 0 \\ & & 0 & \boxed{1} \end{pmatrix} \quad \text{smiley face}$$

ex $\det(A - \lambda I) = (\lambda - 1)^2 (\lambda - 2)^3 = 0$

$\lambda = 1 \Rightarrow \text{alg}(\lambda = 1) = 2$

- $\mu(\lambda = 1) = 2$ $J_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\mu(\lambda = 1) = 1$ $J_{2 \times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\lambda = 2 \Rightarrow \text{alg}(\lambda = 2) = 3$

- $\mu(\lambda = 2) = 3$ $J_{3 \times 3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
- $\mu(\lambda = 2) = 2$ $J_{3 \times 3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ or $J_{3 \times 3} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (worst case)
- $\mu(\lambda = 2) = 1$ $J_{3 \times 3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Case III: Repeated Eigenvalues

Finding J and P

First consider $A_{2 \times 2}$ and J has only one block: (repeated)

$A_{2 \times 2}$ $\text{alg}(\lambda) = 2$, $\mu(\lambda) = 1 \Rightarrow J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

let $P = [\vec{v}^{(1)}, \vec{v}^{(2)}]$

$P^{-1}AP = J \Rightarrow \boxed{AP = PJ}$

$\Rightarrow A\vec{v}^{(1)} = \lambda\vec{v}^{(1)}$

$(A - \lambda I)\vec{v}^{(1)} = \vec{0}$ " $\vec{v}^{(1)}$ is the ordinary eigenvector"

$AP = PJ$

$A[\vec{v}^{(1)}, \vec{v}^{(2)}] = [\vec{v}^{(1)}, \vec{v}^{(2)}] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$\Rightarrow A\vec{v}^{(1)} = \vec{v}^{(1)} + \lambda\vec{v}^{(2)}$

$(A - \lambda I)\vec{v}^{(2)} = \vec{v}^{(1)}$

↓
solve find missing $\vec{v}^{(2)}$ by using $\vec{v}^{(1)}$ not unique pick one solve.

$\vec{v}^{(2)}$ generalized eigenvector obtained by $\vec{v}^{(1)}$

finding $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ (we have two ways)

1st way: (usual way)

① find $\vec{v}^{(1)}$ by looking at $\text{Ker}(A - \lambda I)$

② use obtained $\vec{v}^{(1)}$ in eqn ② $(A - \lambda I) \vec{v}^{(2)} = \vec{v}^{(1)}$ and solve the system for $\vec{v}^{(2)}$

2nd way:

① find $\vec{v}^{(2)}$ by using $(A - \lambda I) \vec{v}^{(2)} = \vec{v}^{(1)}$ [multiply by $(A - \lambda I)$ from left]

$$(A - \lambda I)^2 \cdot \vec{v}^{(2)} = \underbrace{(A - \lambda I) \vec{v}^{(1)}}_{\vec{0}}$$

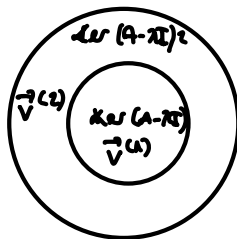
$$\text{so } (A - \lambda I)^2 \cdot \vec{v}^{(2)} = \vec{0} \text{ (hom. system)} \Rightarrow \vec{v}^{(2)} \in \text{Ker}(A - \lambda I)^2 \stackrel{\text{def}}{=} V_{\lambda, 2}$$

$$\text{so choose } \underbrace{\vec{v}^{(1)} \in \text{Ker}(A - \lambda I)}_{V_{\lambda, 1}}^2 \text{ and } \underbrace{\vec{v}^{(2)} \notin \text{Ker}(A - \lambda I)}_{\vec{v}^{(1)}} = V_{\lambda, 1}$$

just to guarantee that $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ are linearly independent.

this produce a chain:

$$\begin{array}{ccc} \text{Ker}(A - \lambda I) & \subsetneq & \text{Ker}(A - \lambda I)^2 \\ \downarrow & & \downarrow \\ \text{define } V_{\lambda, 1} & & V_{\lambda, 2} \end{array}$$



choose $\vec{v}^{(2)} \in V_{\lambda, 2}$ but not in $V_{\lambda, 1}$ then find $\vec{v}^{(1)}$ by using

$$\vec{v}^{(1)} = (A - \lambda I) \vec{v}^{(2)}$$

keep the order

$$\text{then } P = [\vec{v}^{(1)}, \vec{v}^{(2)}]$$

ordinary
eigenvector

generalized
eigenvector

$$\Rightarrow J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (*)$$

$$\text{if we change order in } P = [\vec{v}^{(2)}, \vec{v}^{(1)}] \Rightarrow J = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

ex find J and P for $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = (\lambda-2)^2 = 0$$

$\lambda_1 = 2 \quad \lambda_2 = 2 \quad \text{alg}(\lambda=2) = 2$

eigenvectors:

$$\ker(A - 2I) : (A - 2I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{array}{l} v_1 \text{ lead} \\ v_2 \text{ free} \end{array} \quad 1 \text{ free} \Rightarrow n(\lambda=2) = 1 < \text{alg}(\lambda=2) = 2$$

$$J_{2 \times 2} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad P = (\vec{v}^{(1)}, \vec{v}^{(2)})$$

$\downarrow$
generalized eigenvector

$$\begin{aligned} &\rightarrow v_1 + v_2 = 0 \quad v_1 = -v_2 \\ &v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\ker(A - 2I) = V_{\lambda,1} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

1st way to find $\vec{v}^{(1)}, \vec{v}^{(2)}$

$$\vec{v}^{(1)} \in \ker(A - 2I) \Rightarrow \text{choose } \vec{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{"ordinary eigenvector"}$$

$$\text{for } \vec{v}^{(2)} \text{ solve } (A - 2I)\vec{v}^{(2)} = \vec{v}^{(1)}$$
$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{aligned} -a - b &= -1 \\ a + b &= 1 \end{aligned} \quad a = 1 - b \quad \text{let } b=0, a=1$$

$$\text{so, } \vec{v}^{(2)} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{"generalized eigenvector"}$$

$$P = [\vec{v}^{(1)}, \vec{v}^{(2)}] = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \rightarrow \text{check it}$$

$$J = P^{-1} A P = \underbrace{\frac{1}{-1} \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}_P \quad (\text{matrix multiplication sign})$$

Now we shall see the 2nd way of finding $\vec{v}^{(1)}, \vec{v}^{(2)}$

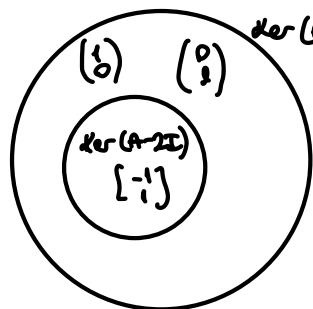
2nd way finding $\vec{v}^{(1)}$, $\vec{v}^{(2)}$

first find $\vec{v}^{(2)}$ by looking $\ker(A - \lambda I)^2 = \ker(A - 2I)^2$

$$(A - 2I)^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

choose $\vec{v}^{(2)} \in \ker(A - 2I)^2 = V_{\lambda, 2}$ and $\vec{v}^{(2)} \notin \ker(A - 2I) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$-v_1 + v_2 = 0$$



$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

wlog choose $\vec{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V_{\lambda, 2}$ but not in $V_{\lambda, 1}$

then find $\vec{v}^{(1)} = (A - 2I) \vec{v}^{(2)}$

$$\vec{v}^{(1)} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{so, } P = (\vec{v}^{(1)}, \vec{v}^{(2)}) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Note that we can choose $\vec{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as well, this time we have

$$\vec{v}^{(1)} = (A - 2I) \vec{v}^{(2)} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \rightarrow \text{check exercise}$$

ex find P and J for $A = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix}$

$$\text{eigenvalues } \begin{pmatrix} -7-\lambda & 1 \\ -4 & -3-\lambda \end{pmatrix} \Rightarrow (\lambda + 5)^2 = 0 \quad \lambda_1 = \lambda_2 = -5 \quad \text{alg}(\lambda = -5) = 2 \text{ (repeated)}$$

$$\text{eigenvectors } \ker(A + 5I) = V_{\lambda, 1}$$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 \text{ lead} \\ v_2 \text{ free} \end{matrix} \rightarrow 1 \text{ free} \quad m(\lambda = -5) = 1 < 2 = \text{alg}(\lambda = -5)$$

$$2v_1 - v_2 = 0$$

$$\text{so } J = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix} \quad \ker(A + 5I) = \left\{ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid 2v_1 - v_2 = 0 \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

1. find $\vec{v}^{(2)}$ $\ker (A+5I)^2$ | $(A+5I)^2 \vec{v}^{(1)} = \vec{0}$
 $\Downarrow$
 $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(A+5I)^2 = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad a, b \text{ free}$$

$$\Rightarrow \vec{v}^{(2)} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \ker (A+5I)^2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

choose $\vec{v}^{(2)} \in V_{\lambda,2} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ but not in $V_{\lambda,1} = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

wlog choose $\vec{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \ker (A+5I)^2 \notin \ker (A+5I) = \{ \vec{v} \mid 2v_1 - v_2 = 0 \}$

then find $\vec{v}^{(1)} = (A+5I) \vec{v}^{(2)} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$$\therefore p = (\vec{v}^{(1)}, \vec{v}^{(2)}) = \begin{pmatrix} -2 & 1 \\ -4 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix}$$

$\downarrow$ $\downarrow$
 ordinary generalized

exercise find $\vec{v}^{(1)}, \vec{v}^{(2)}$ by using 1st way.

Question what is the soln of

$$\vec{x}' = A \vec{x} \Rightarrow \vec{x}' = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix} \vec{x}$$

$$\vec{x}_1 = \underbrace{e^{\lambda t}}_{\checkmark} \vec{v}^{(1)}, \quad \underbrace{e^{\lambda t}}_x \vec{v}^{(2)} = \vec{x}_2$$

how can we write the soln? we need to use fundamental matrix

actually exponential matrix e^{At} so what is e^{At} ?

Remember: if $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are fundamental solns to hom. system $\vec{x}' = A \vec{x}$

the matrix $\Psi = [\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n]$ is called fund. matrix for $\vec{x}' = A \vec{x}$

then the general soln to system is $\vec{x}_{\text{gen}} = \Psi \vec{c} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n$

Then if Ψ is a fundamental matrix for (H) $\vec{x}' = A \vec{x}$ then

$\Psi(t)P$ is also a fundamental matrix of (H) for any non singular (invertible) matrix P ($\det(P) \neq 0$)

pf let $P = \Psi^{-1}(t_0) \Rightarrow \Psi(t)P = \Psi(t) \cdot \Psi^{-1}(t_0) = \Phi(t)$

st. $\Phi(t_0) = \Psi(t_0) \Psi^{-1}(t_0) = I$ (identity)

that is $\Phi(t) = [\vec{y}_1(t), \vec{y}_2(t) \dots \vec{y}_n(t)]$ where each column

$$\vec{y}_j(t_0) = e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow j\text{th row}$$

then the soln of IVP $\vec{x}' = A \vec{x}$, $\vec{x}(t_0) = \vec{x}^0$ becomes

$$\vec{x}(t) = \Phi(t) \vec{x}^0 \quad \hookrightarrow \text{given in question}$$

Remark we usually take $t_0 = 0$

Defn $\Phi(t) = \Psi(t) \cdot \Psi^{-1}(t_0)$ with $\Phi(t_0) = I$ is called "the special fund. matrix"

for (H) $\vec{x}' = A \vec{x}$. Moreover, $\Phi(t)$ is unique. Here, $\Psi(t)$ is any fundamental matrix for (H) .

ex $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$ find $\Psi(t)$ and $\Phi(t)$ for $t_0 = 0$

eigenvalues $\lambda^2 - 1 = 0$ $\lambda_1 = 1$ $\lambda_2 = -1$ real, distinct, $\underline{e^{\lambda_1 t} \vec{v}^{(1)}}$, $\underline{e^{\lambda_2 t} \vec{v}^{(2)}}$

eigenvectors

$\lambda_1 = 1$ $\ker(\lambda - I)$ $((\lambda - I) \vec{v}^{(1)} = 0)$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 \text{ lead} \\ v_2 \text{ free} \end{matrix} \quad v_1 - v_2 = 0 \quad v_1 = v_2 \quad \vec{v}^{(1)} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\ker(A - I) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\vec{x}_1(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

$$\lambda = -1 \quad \ker(\lambda + I) \quad ((A+I)\vec{v} = \vec{0})$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \quad 3v_1 - v_2 = 0 \rightarrow v_2 = 3v_1 \quad \ker(A+I) = \left\{ \vec{v} \mid 3v_1 - v_2 = 0 \right\}$$

$$= \text{span} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

$$\vec{x}_2(t) = e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-3t} \end{pmatrix}$$

$$= \varphi(t) = [\vec{x}_1, \vec{x}_2] = \begin{pmatrix} e^t & e^{-t} \\ e^t & e^{-3t} \end{pmatrix} \quad \text{any fundamental matrix}$$

$$\Phi(t) = \varphi(t) \varphi^{-1}(t_0) \quad \Big|_{t_0=0}$$

$$\text{find } \varphi^{-1}(0) : \varphi(0) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \rightarrow \varphi^{-1}(0) = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\text{then the special fund. matrix } \Phi = \varphi(t) \cdot \varphi^{-1}(0)$$

$$= \begin{pmatrix} e^t & e^{-t} \\ e^t & e^{-3t} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - e^{-3t} & -e^t + 3e^{-3t} \end{pmatrix}$$

$$\Phi(0) = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Question: what the hell is of $\Phi(t)$?

Answer: it's a god damn soln of IVP
$$\begin{cases} \Phi'(t) = A \Phi(t) \\ \Phi(0) = I \end{cases}$$

now look at e^{At} :

e^{At} is called matrix exponential where A is constant matrix.

Remember
$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \frac{at}{1} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots \quad a \in \mathbb{R}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \dots \quad \text{converges} \quad \forall t \in \mathbb{R}$$

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{(At)^k}{k!}$$

check
$$(e^{At})' = \frac{d}{dt} (e^{At}) = \sum_{k=1}^{\infty} A k \frac{(At)^{k-1}}{(k-1)!} = A \underbrace{\sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!}}_{\sum_{k=0}^{\infty} \frac{(At)^k}{k!}} = A e^{At}$$

$$(e^{At})' = A \cdot e^{At}$$

This means that e^{At} satisfy (4) $\vec{x}' = A\vec{x}$

and moreover, $e^{At} \Big|_{t=0} = \left(I + \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \right) \Big|_{t=0} = I$

That is e^{At} is the sol for $\vec{x}' = A\vec{x}$ with $e^{At} \Big|_{t=0} = I$

Conclusion: $e^{At} = \Phi(t)$ where $\Phi(0) = I$

That is, the special fundamental matrix is nothing but equal to $e^{At} = \Phi(t)$

Def: the exponential matrix e^{At} where A is constant is the special fundamental matrix unique

for homogenous system $\vec{x}' = A\vec{x}$ which satisfy $e^{At} \Big|_{t=0} = I$

That is, $e^{At} = \Phi(t) = \Psi(t) \Psi^{-1}(0)$ where $\Psi(t)$ is any fundamental matrix.

Ex $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$. find $\Psi(t)$, $\Phi(t)$, $e^A = e^{At} \Big|_{t=1}$ not unique

eigenvalues: $(1-\lambda)^2 - 4 = (\lambda-3)(\lambda+1) = 0$
 $\lambda=3, \lambda=-1$

eigenvectors: $\underline{\lambda=3}$ $\ker(A-3I) : (A-3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow -2v_1 + v_2 = 0 \quad \begin{matrix} v_1 \text{ lead} \\ v_2 \text{ free} \end{matrix} \quad \ker(A-3I) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid -2v_1 + v_2 = 0 \right\}$$
$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

so, $\vec{x}_1(t) = e^{\overset{\lambda=3}{3t}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{\lambda=-1}$ $\ker(A+I) : (A+I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 2u_1 + u_2 = 0 \quad \begin{matrix} u_1 \text{ lead} \\ u_2 \text{ free} \end{matrix} \Rightarrow \ker(A+I) = \left\{ \vec{u} \mid 2u_1 + u_2 = 0 \right\}$$
$$= \text{span} \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

so $\vec{x}_2(t) = e^{\overset{\lambda=-1}{-t}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Thus $\Psi(t) = [\vec{x}_1, \vec{x}_2] = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$, $\Psi(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ $\Psi^{-1}(0) = \frac{-1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$

$$\Phi(t) = e^{At} = \Psi(t) \Psi^{-1}(0) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \left(\frac{-1}{4} \right) \cdot \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} = \frac{1}{4} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix} = \Phi(t)$$

observe that $\Phi(0) = I$

$$e^A = e^{At} \Big|_{t=1} = \frac{1}{4} \begin{bmatrix} 2e^3 + 2e^{-1} & e^3 - e^{-1} \\ 4e^3 - 4e^{-1} & 2e^3 + 2e^{-1} \end{bmatrix}$$

Thm: if $\Psi(t)$ is a fundamental matrix then $\Psi(t) \cdot P$ is also a fundamental matrix

where P is invertible

15 Nisan 2024

connection between eigenvectors and e^{At}

e^{At} is the special fundamental matrix for $\vec{x}' = A\vec{x}$ ($\vec{H}$)

how can we find e^{At} without calculating $\Phi(t) = \Psi(t) \Psi^{-1}(0)$

1) A is diagonal

$$A = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A^1 = D^1 = \begin{pmatrix} \lambda_1^1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^1 \end{pmatrix}$$

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + A^2 \frac{t^2}{2!} + \dots + = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + t \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} & & 0 \\ & \ddots & \\ 0 & & 1 + \lambda_n t + \frac{\lambda_n^2 t^2}{2!} \end{bmatrix} \end{aligned}$$

$$\Rightarrow = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} = e^{At}, \quad A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then soln to $\vec{x}' = A\vec{x}$ is $\vec{x}_{gen} = e^{At} \vec{c} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + c_n e^{\lambda_n t} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

2) A is diagonalizable

$$\exists \text{ invertible } P \text{ st } P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad P^{-1}AP = D \Leftrightarrow A = PDP^{-1}$$

$$\Rightarrow e^{At} = e^{PDP^{-1}t} = I$$

$$= I + PDP^{-1}t + \underbrace{(PDP^{-1})^2}_{PD^2P^{-1}} \frac{t^2}{2!} + \underbrace{(PDP^{-1})^3}_{PD^3P^{-1}} \frac{t^3}{3!} + \dots$$

$$= P \left(\underbrace{I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!}}_{e^{Dt}} \right) \cdot P^{-1} \Rightarrow P e^{Dt} P^{-1} = e^{At} \quad ?$$

where $e^{Dt} = \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t})$

$$e^{At} = e^{PDP^{-1}t} = P e^{Dt} P^{-1}$$

then the soln to $\vec{x}' = A\vec{x}$ is $\vec{x}_{gen} = e^{At} \vec{c} = P e^{Dt} P^{-1} \vec{c}$

and we can write gen soln also by using this saying $e^{At} P$ is also fund. matrix:

$$\vec{x}_{gen} = e^{At} P \vec{c} = P e^{Dt} \cdot P^{-1} P \vec{c} = P e^{Dt} \vec{c}$$

where P is invertible matrix obtained taking each eigenvector as a column.

$$P = [\vec{v}^{(1)} \quad \vec{v}^{(2)} \quad \dots \quad \vec{v}^{(n)}]$$

3) A is NOT diagonalizable

in this case A can be written in Jordan form. That is $\exists$ invertible P st

$$P^{-1}AP = J \quad (J: \text{Jordan}) \Rightarrow A = PJP^{-1}$$

$$e^{At} = e^{PJP^{-1}t} = P \cdot \overset{?}{e^{Jt}} \cdot P^{-1}$$

Then (exponentiating a matrix in Jordan form) if we consider 1 Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda & 1 \end{bmatrix} = e^{Jt} = e^{\lambda t} \cdot \begin{bmatrix} 1 & t & t^2/2! & \dots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \dots & t^{n-2}/(n-2)! \\ \vdots & & 1 & t & \dots & t \\ 0 & \dots & \dots & \dots & 1 \end{bmatrix}$$

ex $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\lambda = 2 \Rightarrow e^{Jt} = e^{2t} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

ex $J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, $\lambda = -1 \Rightarrow e^{Jt} = e^{-t} \cdot \begin{pmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

ex $J = \left(\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 0 & 3 \end{array} \right)$ $J_1 = [3]$
 $J_2 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, $e^{Jt} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$$e^{Jt} = \left(\begin{array}{c|cc} e^{J_1 t} & 0 & 0 \\ \hline 0 & e^{J_2 t} \end{array} \right) = \left(\begin{array}{c|cc} e^{J_1 t} & 0 & 0 \\ \hline 0 & e^{J_2 t} & t e^{J_2 t} \\ 0 & 0 & e^{J_2 t} \end{array} \right)$$

then soln to $\vec{x}' = A\vec{x}$ where $A = PJP^{-1}$

$$\vec{x}_{gen} = e^{At} \vec{c} = P e^{Jt} P^{-1} \vec{c} \quad \text{OR} \quad \vec{x}_{gen} = e^{At} P \vec{z} = P e^{Jt} \vec{z}$$

ex $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$

- find J and P ✓ done prev lecture
- find $\vec{x}_{gen}$
- Draw phase portrait

$$\det(A - \lambda I) \Rightarrow \lambda_1 = \lambda_2 = 2$$

$$\text{alg}(\lambda=2) = 2$$

eigenvectors: $\ker(\lambda - 2I) = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ v_1 lead v_2 free

of block in J
 $M(\lambda=2) = 1 < 2 = \text{alg}(\lambda=2)$

$$\ker(A - 2I) = \left\{ \vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mid v_1 + v_2 = 0 \right\} \Rightarrow \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$v_1 + v_2 = 0$ you can choose v_1 as const of v_2

for generalized eigenvector $\vec{v}_2$ look for $\ker(A - 2I)^2$

$$(A - 2I)^2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad v_1, v_2 \text{ free.} \Rightarrow \ker(A - 2I)^2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Choin: $V_{\lambda,1} = \ker(A - 2I) \subsetneq V_{\lambda,2} = \ker(A - 2I)^2$

choose $\vec{v}_2$ st $\vec{v}_2 \in \ker(A - 2I)^2$ and $\vec{v}_2 \notin \ker(A - 2I)$

WLOG choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V_{\lambda,2} \not\in V_{\lambda,1}$ now find $\vec{v}_1$ using $\vec{v}_2$: $\vec{v}_1 = (A - 2I)\vec{v}_2$

$$\vec{v}_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = (\vec{v}_1 \vec{v}_2) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}_{v_1 \ v_2}, \quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad e^{Jt} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

"ordinary eigenvector"

Handwritten notes and diagrams illustrating the Jordan form and the exponential of a Jordan matrix.

Diagram 1: $J = \begin{bmatrix} \boxed{2} & & \\ & \boxed{\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}} & \\ & & \boxed{\begin{smallmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{smallmatrix}} \end{bmatrix}$

Diagram 2: $e^{Jt} = \begin{bmatrix} e^{J_1 t} & & \\ & e^{J_2 t} & \\ & & e^{J_3 t} \end{bmatrix}$

Diagram 3: $e^{J_1 t} = e^{2t}$

Diagram 4: $e^{J_2 t} = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

Diagram 5: $e^{J_3 t} = e^{-t} \begin{bmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

Diagram 6: $e^{Jt} = \begin{bmatrix} e^{2t} & t e^{2t} & t^2 e^{2t}/2! \\ 0 & e^{2t} & t e^{2t} \\ 0 & 0 & e^{-t} \end{bmatrix}$

$$\vec{x}_{gen} = e^{At} \vec{z} \quad \text{OR} \quad \underline{e^{At} P \vec{z}}$$

$$\begin{aligned} \vec{x}_{gen} &= e^{At} P \vec{z} = P \underbrace{e^{Jt} P^{-1} P}_{I} \vec{z} \\ &= P e^{Jt} \vec{z} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \vec{z} \end{aligned}$$

$$\vec{x}_{gen} = c_1 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -t+1 \\ t \end{pmatrix} = e^{2t} \begin{bmatrix} -1 & -t+1 \\ 1 & t+1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= c_1 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -t \\ t \end{bmatrix}$$

$$\vec{x}_{gen} = c_1 \boxed{e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}} + c_2 \boxed{e^{2t} \left\{ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}$$

$\vec{x}_1(t)$

soln coming from
ordinary eigenvector

$$\vec{x}_2(t) = c_2 e^{2t} \left\{ t \vec{v}_1 + \vec{v}_2 \right\}$$

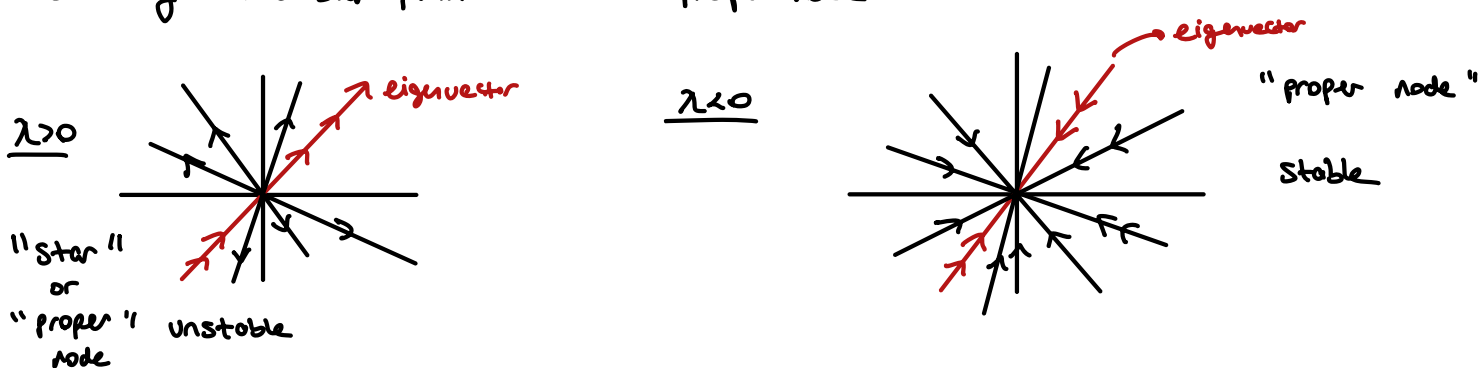
soln coming from
generalized eigenvector

Drawing Phase Portrait :

A has 2 repeated eigenvalues.

Case I ☺ $\text{alg}(\lambda) = \text{m}(\lambda) \Rightarrow$ 2 linearly independent eigenvectors.

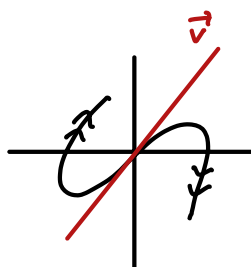
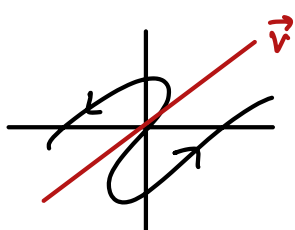
So origin is a star point or a proper node



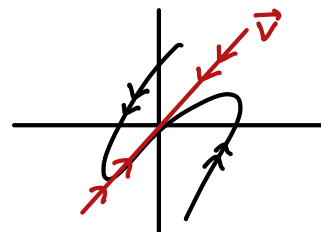
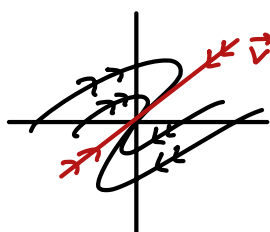
Case II ☹ $\text{m}(\lambda) = 1 < 2 = \text{alg}(\lambda) \Rightarrow$ only one linearly independent eigenvector

Origin is called an "improper node"

$\lambda > 0$ "unstable"



$\lambda < 0$ "stable improper node"



to draw phase portrait

1) Draw ordinary eigenvector. Decide about direction of arrow by looking at sign of λ

$$\lambda > 0 \Rightarrow \text{outward}$$

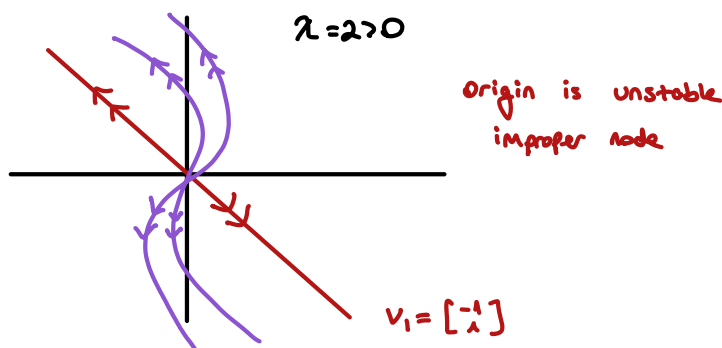
$$\lambda < 0 \Rightarrow \text{towards origin}$$

2) for other solns pick a point like

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and subs in system}$$

$$\vec{x}' = A\vec{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ "NOT GOOD" } \quad \text{so try } \vec{e}_2 \quad A\vec{e}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\vec{x} = \vec{e}_1$ could not decide $\begin{matrix} \uparrow \\ \downarrow \end{matrix}$?



ex find P and J for $A = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix}$

eigenvalues $\begin{pmatrix} -7-\lambda & 1 \\ -4 & -3-\lambda \end{pmatrix} \Rightarrow (\lambda+5)^2 = 0 \quad \lambda_1 = \lambda_2 = -5 \quad \text{alg}(\lambda = -5) = 2 \text{ (repeated)}$

eigenvectors $\text{ker}(A+5I) = v_{\lambda,1}$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 \text{ lead} \\ v_2 \text{ free} \end{matrix} \rightarrow 1 \text{ free} \quad m(\lambda = -5) = 1 < 2 = \text{alg}(\lambda = -5)$$

so $J = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix} \quad \text{ker}(A+5I) = \left\{ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid 2v_1 - v_2 = 0 \right\}$

$2v_1 - v_2 = 0$

$= \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

now

$$P = (\vec{v}^{(1)} \quad \vec{v}^{(2)}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$e^{Jt} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = e^{-5t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\vec{x}_{gen} = e^{At} \vec{c} = e^{P^{-1}JP} e^{Jt} \vec{c} = P e^{Jt} \underbrace{P^{-1}\vec{c}}_{\vec{z}} = P e^{Jt} \vec{z}$$

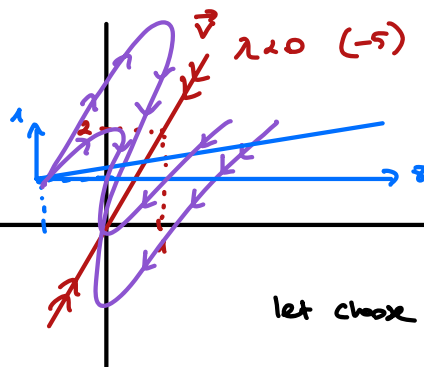
$$\text{so } \vec{x}_{gen} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} e^{-5t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \vec{c}$$

$$= c_1 e^{-5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} t \\ 2t+1 \end{pmatrix}$$

$$\left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\underbrace{\quad}_{\vec{v}^{(1)}} \quad \underbrace{\quad}_{\vec{v}^{(2)}}$$

(all sol go to origin
 $t \rightarrow \infty$
 (te^{-5t}))



$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
 origin is stable
 improper node

let choose $\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{e}_2$

$$A \vec{e}_2 = \begin{pmatrix} -3 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

when A is 3×3 $\vec{x}' = A \vec{x}$ $\text{alg}(\lambda) = 3$, $\mu(\lambda) = 1$

consider 1 jordan Block $\dot{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}_{3 \times 3}$

$$P = [\vec{v}^{(1)} \vec{v}^{(2)} \vec{v}^{(3)}]$$

$$P^{-1} A P = \dot{J} \Rightarrow A P = P \dot{J}$$

$$A [\vec{v}^{(1)} \vec{v}^{(2)} \vec{v}^{(3)}] = [\vec{v}^{(1)} \vec{v}^{(2)} \vec{v}^{(3)}] \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\textcircled{1} A \vec{v}^{(1)} = \lambda \vec{v}^{(1)}$$

$$\textcircled{2} A \vec{v}^{(2)} = \lambda \vec{v}^{(2)} + \vec{v}^{(1)}$$

$$\textcircled{3} A \vec{v}^{(3)} = \lambda \vec{v}^{(3)} + \vec{v}^{(2)}$$

$$\textcircled{1} (A - \lambda I) \vec{v}^{(1)} = \vec{0} \quad \vec{v}^{(1)}: \text{ordinary eigenvector}$$

$$\textcircled{2} (A - \lambda I) \vec{v}^{(2)} = \vec{v}^{(1)} \quad \vec{v}^{(2)}: \text{generalized eigenvector obtained from } \vec{v}^{(1)}$$

$$\textcircled{3} (A - \lambda I) \vec{v}^{(3)} = \vec{v}^{(2)} \quad \vec{v}^{(3)}: \text{generalized eigenvector obtained from } \vec{v}^{(2)}$$

2nd way (17 Nisan 2024)

$$\textcircled{1} (A - \lambda I) \vec{v}^{(1)} = \vec{0}$$

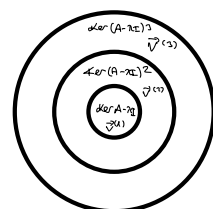
$$\textcircled{2} (A - \lambda I) \vec{v}^{(2)} = \vec{v}^{(1)} \quad \begin{matrix} \text{multiply with} \\ A - \lambda I \text{ from left} \end{matrix} \quad (A - \lambda I)^2 \vec{v}^{(2)} = (A - \lambda I) \vec{v}^{(1)} = \vec{0} \Rightarrow \vec{v}^{(2)} \in \ker(A - \lambda I)^2$$

$$\textcircled{3} (A - \lambda I) \vec{v}^{(3)} = \vec{v}^{(2)} \quad \begin{matrix} \text{multiply with} \\ (A - \lambda I)^2 \end{matrix} \quad (A - \lambda I)^3 \vec{v}^{(3)} = (A - \lambda I)^2 \vec{v}^{(2)} = \vec{0} \Rightarrow \vec{v}^{(3)} \in \ker(A - \lambda I)^3$$

construct a chain:

$$\underbrace{\ker(A - \lambda I)}_{\vec{v}_{\lambda,1}} \subsetneq \underbrace{\ker(A - \lambda I)^2}_{\vec{v}_{\lambda,2}} \subsetneq \underbrace{\ker(A - \lambda I)^3}_{\vec{v}_{\lambda,3}}$$

choose $\vec{v}^{(3)}$ st $\vec{v}^{(3)} \in \ker(A - \lambda I)^3$
 $\vec{v}^{(3)} \notin \ker(A - \lambda I)^2$



construct ③ $\vec{v}^{(3)} = (A - \lambda I) \vec{v}^{(2)}$ [subs. chosen $\vec{v}^{(2)}$] matrix mult. not solve system

② $\vec{v}^{(1)} = (A - \lambda I) \vec{v}^{(2)}$

$\vec{v}^{(2)} \in V_{\lambda,2}$ but $\vec{v}^{(2)} \notin V_{\lambda,1}$ and $\vec{v}^{(1)} \in V_{\lambda,1}$ be careful

Generalization $A_{n \times n}$: consider only one Jordan Block

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots \\ 0 & \lambda & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda \end{pmatrix} \quad P^{-1}AP = J \quad AP = PJ \quad P = (\vec{v}^{(1)} \vec{v}^{(2)} \dots \vec{v}^{(n)})$$

$$A \vec{v}^{(1)} = \lambda \vec{v}^{(1)}$$

$$(A - \lambda I) \vec{v}^{(1)} = 0$$

$$A \vec{v}^{(2)} = \lambda \vec{v}^{(2)} + \vec{v}^{(1)}$$

Rewrite $(A - \lambda I) \vec{v}^{(2)} = \vec{v}^{(1)} \Rightarrow (A - \lambda I)^2 \vec{v}^{(2)} = (A - \lambda I) \vec{v}^{(1)} = 0$

$\vdots$

$$A \vec{v}^{(n)} = \lambda \vec{v}^{(n)} + \vec{v}^{(n-1)} \Rightarrow (A - \lambda I)^n \vec{v}^{(n)} = (A - \lambda I)^{n-1} \vec{v}^{(n-1)} = 0 \quad \vec{v}^{(n)} \in \ker(A - \lambda I)^n$$

construct a chain

$$\ker(A - \lambda I) \subsetneq \ker(A - \lambda I)^2 \subsetneq \dots \subsetneq \ker(A - \lambda I)^n$$

$$\vec{v}^{(1)} \in \ker(A - \lambda I)$$

construct

$$\vec{v}^{(n-1)} = (A - \lambda I) \vec{v}^{(n)}$$

$$\vec{v}^{(n-2)} = (A - \lambda I) \vec{v}^{(n-1)}$$

$$\vdots$$

$$\vec{v}^{(1)} = (A - \lambda I) \vec{v}^{(2)}$$

ex $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$ a) find P, J b) solve $\vec{x}' = A\vec{x}$ (1st way exercise)

eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow -(\lambda - 2)^3 = 0 \quad \lambda = 2 \quad \text{alg}(\lambda = 2) = 3$

eigenvectors $(A - 2I) \vec{v}^{(1)} = \vec{0} \quad \vec{v}^{(1)} \in \ker(A - 2I)$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 & v_2 & \text{lead} \\ & v_3 & \text{free} \end{matrix} \Rightarrow \mu(\lambda = 2) = 1 \quad \# \text{ of jordan block. } J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\ker(A - 2I) = \left\{ \vec{v}^{(1)} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid \begin{matrix} -v_1 + v_2 + v_3 = 0 \\ v_2 + v_3 = 0 \end{matrix} \right\} \Rightarrow \left\{ \vec{v}^{(1)} \mid v_1 = 0, v_2 = -v_3 \right\} \Rightarrow \text{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\} \underline{V_{\lambda,1}}$$

1st way: ① $(A - 2I) \vec{v}^{(2)} = \vec{v}^{(1)}$ solve system for $\vec{v}^{(2)}$

then ② $(A - 2I) \vec{v}^{(3)} = \vec{v}^{(2)}$ solve system for $\vec{v}^{(3)}$

} exercise

2nd Way :

compute $V_{\lambda,2} = \text{Ker}(A-2I)^2$ $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ v_1 lead v_2, v_3 free

$$\text{Ker}(A-2I)^2 = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} : v_1 - v_2 - v_3 = 0 \right\} \Rightarrow \left\{ \begin{pmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{pmatrix} \right\} \Rightarrow \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

compute $(A-2I)^3$ must be zero $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ v_1, v_2, v_3 free

$$\text{Ker}(A-2I)^3 = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, v_1 \text{ free} \right\} \Rightarrow \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

pick one $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v}^{(3)} \in V_{\lambda,3}$ and not in $V_{\lambda,2}$ Should not satisfy $v_1 - v_2 - v_3 = 0$

then find $\vec{v}^{(2)} = (A-2I)\vec{v}^{(3)} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ (must satisfy $v_1 - v_2 - v_3 = 0$)

then find $\vec{v}^{(1)} = (A-2I)\vec{v}^{(2)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

then $P = (\vec{v}^{(1)} \ \vec{v}^{(2)} \ \vec{v}^{(3)}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix}$ $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\vec{x}_{\text{gen}} = \psi(t) \cdot c$ or $e^{At} c$ * need to find P^{-1} we don't wanna deal with it but in exam this will be asked.

$$\begin{aligned} &\Downarrow \\ &e^{At} P \vec{c} \\ &P e^{Jt} \vec{c} \end{aligned}$$

$$e^{Jt} = e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{x}_{\text{gen}} = P e^{Jt} \vec{c} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & t e^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t} & t e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \boxed{c_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}} + \boxed{c_2 e^{2t} \begin{pmatrix} -1 \\ -t+2 \\ t-3 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} -t+1 \\ -t^2/2+t \\ t^2/2-2t \end{pmatrix}}$$

soln coming from generalized eigenvector

soln comes from ordinary eigenvector $\vec{v}^{(1)}$

$$t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$\vec{v}^{(1)} \quad \vec{v}^{(2)}$

$$\frac{t^2}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}^{(1)} \quad \vec{v}^{(2)} \quad \vec{v}^{(3)}$

ex Solve $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

eigenvalues $(A - \lambda I) = (-3 - \lambda)^3 = 0$ $\lambda = -3$ $\text{alg}(\lambda) = 3$ $J_{3 \times 3}$

eigenvectors (find $\ker(A - \lambda I)$)

$\begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ v_1, v_3 free $\rightarrow 2$ j blocks $= m(\lambda) = 2 = 2$ lin indep eigenvectors
 v_2 leading

$\ker(A + 3I) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid 2v_2 - v_3 = 0 \right\} = \left[\begin{matrix} v_1 \\ 1/2 v_3 \\ v_3 \end{matrix} \right] = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$
 we don't know if gen eigenvector comes from which? because of 2 j blocks.
 $v_1 = 1, v_3 = 0$ generated
 $v_1 = 0, v_3 = 1$ generated

$J_{3 \times 3} = \left(\begin{array}{ccc|ccc} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \end{array} \right)$ OR $\left(\begin{array}{ccc|ccc} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \end{array} \right)$ $P = (v^{(1)}, v^{(3)}, v^{(2)})$
 related separated

$P = (v^{(1)}, \underbrace{v^{(2)}, v^{(3)}}_{\text{generalized}}, v^{(2)})$
 separated related

Remark: now it is unclear which eigenvector leads to generalized eigenvector $\vec{v}^{(3)}$.

That is $(A - \lambda I) \vec{v}^{(3)} = \vec{v}^{(1)}$
 $(A - \lambda I) \vec{v}^{(3)} = \vec{v}^{(2)}$ which system to solve is unclear but we can try.

but instead use 2nd way for clarity. that is start by choosing by finding $\ker(A + 3I)^2$

choose $\vec{v}^{(3)}$ st $\vec{v}^{(3)} \in \ker(A + 3I)^2$

$\vec{v}^{(3)} \notin \ker(A + 3I)$

find $\ker(A + 3I)^2$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
 cannot close ok ok

choose $\vec{v}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \ker(A + 3I)^2$
 $\notin \ker(A + 3I)$

$\in \ker(A + 3I)$

then find ordinary eigenvector $\vec{v}^{(1)}$ (we said it) related to $\vec{v}^{(3)} \Rightarrow \vec{v}^{(1)} = (A + 3I) \vec{v}^{(3)} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

NOTE $\vec{v}^{(1)}? (A + 3I) \vec{v}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ can't be eigenvector not true

So how can we obtain $\vec{v}^{(1)}$? choose $\vec{v}^{(1)} \in \ker(A + 3I)$ and lin indep with $\vec{v}^{(2)}$ ($\vec{v}^{(3)}$): $\vec{v}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Now $\textcircled{*} P = (v^{(1)} \boxed{v^{(2)} \ v^{(3)}})$ $J = \left(\begin{array}{c|cc} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{array} \right)$

$\textcircled{**} P = (\boxed{v^{(2)} \ v^{(3)}} \ v^{(1)})$ $J = \left(\begin{array}{c|cc} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{array} \right)$

$\vec{x}_g$ with $\textcircled{*}$

$\vec{x}_g = \mathcal{L}(t) \vec{c} = P e^{Jt} \vec{c}$

$J = \left(\begin{array}{c|cc} -3 & & \\ \hline & -3 & 1 \\ & 0 & -3 \end{array} \right)$ $e^{Jt} = \left(\begin{array}{c|cc} e^{j_1 t} & & \\ \hline & e^{j_2 t} & \\ & & \end{array} \right)$

$e^{j_1 t} = [e^{-3t}]$

$= e^{-3t} \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & t \\ 0 & 0 & 1 \end{array} \right)$

$e^{j_2 t} = e^{-3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\vec{x}_g = P e^{Jt} \vec{c}$

$= \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} e^{-3t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \vec{c}$

$\vec{x}_g = c_1 e^{-3t} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{v^{(1)}} + c_2 e^{-3t} \underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}_{v^{(2)}} + c_3 e^{-3t} \left[t \underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}_{v^{(2)}} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v^{(3)}} \right]$

Soln from ordinary eigenvector

Soln from generalized eigenvector

exercise:

1) find A^{254} if $A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}$

check diagonalizable if not check jordan form

$A = P D P^{-1} \Rightarrow A^{254} = P D^{254} P^{-1}$

if jordan,

$A = P J P^{-1} \Rightarrow A^{254} = P J^{254} P^{-1}$

Remark $J^n = \begin{bmatrix} \lambda^n & n \cdot \lambda^{n-1} & \frac{n(n-1)}{2} \lambda^{n-2} \\ 0 & \lambda^n & n \cdot \lambda^{n-1} \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & \lambda^n \end{bmatrix}$

$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

$J^n = \left[\begin{array}{cc|c} \lambda^n & n \cdot \lambda^{n-1} & 0 \\ 0 & \lambda^n & 0 \\ \hline 0 & 0 & \lambda^n \end{array} \right]$

2) find e^{At} for $\vec{x}' = A x$

$A = \begin{bmatrix} 5 & 0 & 4 \\ -6 & 1 & -1 \\ -6 & 0 & -5 \end{bmatrix}$

Hint: $\lambda_1 = -1$
 $\lambda_{2,3} = 1$

$J = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right]$

Non homogeneous Systems

(N) $\vec{x}' = A\vec{x} + \vec{g}(t)$ where A : constant

$\vec{x}_g = \vec{x}_h + \vec{x}_p$

(H) $\vec{x}' = A\vec{x}$ Soln. corresponding to homogeneous eqn (soln for homogeneous part) Complimentary soln. We know how to obtain $\vec{x}_h$ (or $\vec{x}_c$)

particular soln. to $\vec{x}' = A\vec{x} + \vec{g}(t)$ "not familiar" To find $\vec{x}_p$ we'll use variation of parameter technique (VOP)

Variation of parameter (VOP) technique:

We know that $\vec{x}_h = \Psi(t)\vec{c}$ where $\Psi(t) = [\vec{x}_1 \ \vec{x}_2 \ \dots \ \vec{x}_n]$ with $\vec{x}_i$: linearly independent solns. $i=1, \dots, n$ for hom. syst. $\vec{x}' = A\vec{x}$.

Now assume that $\vec{x}_p = \Psi(t)\vec{u}(t)$

Substitute $\vec{x}_p = \Psi(t)\vec{u}(t)$ in (N) $\vec{x}' = A\vec{x} + \vec{g}(t)$

$\vec{x}_p' = \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t)$

$\vec{x}_p' = A\vec{x}_p + \vec{g}(t)$

$\Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) = A\Psi(t)\vec{u}(t) + \vec{g}(t)$

$\Psi(t)\vec{u}'(t) = \vec{g}(t)$

$\vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t)$

$\vec{u}(t) = \int \Psi^{-1}(t)\vec{g}(t) dt$

$\vec{x}_p = \Psi(t)\vec{u}(t) = \Psi(t) \int \Psi^{-1}(t)\vec{g}(t) dt$

so, the general soln of $\vec{x}' = A\vec{x} + \vec{g}(t)$

$$\vec{x}_g = \vec{x}_h + \vec{x}_p = \Psi(t) \cdot \vec{c} + \Psi(t) \int \Psi^{-1}(t) \vec{g}(t) dt$$

ex $\vec{x}' = \underbrace{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}}_A \vec{x} + \underbrace{\begin{bmatrix} e^t \\ t \end{bmatrix}}_{\vec{g}(t)}$

a) Solve $\vec{x}' = A\vec{x}$, draw phase portrait

b) Solve $\vec{x}' = A\vec{x} + \vec{g}(t)$

a) eigenvalues

$$\det(A - \lambda I) = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\lambda = -1 \text{ (ker } \lambda + I)$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \text{lead} & \text{free} \\ 3v_1 - v_2 = 0 \end{matrix}$$

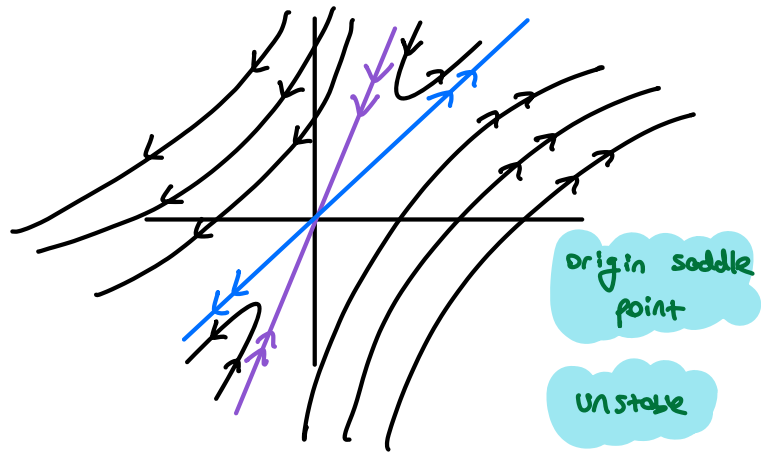
$$\text{ker}(A + I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \Rightarrow x_2(t) = e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{matrix} \text{lead} & \text{free} \\ u_1 - u_2 = 0 \end{matrix}$$

$$\text{ker}(A - I) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{aligned}\vec{x}_h &= c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \underbrace{\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{c}}\end{aligned}$$



b) VOP

assume $\vec{x}_p = \Psi(t) \vec{\mu}(t)$ (subs in (N))

$$\vec{x}_p' = A \vec{x}_p + \vec{g}(t)$$

$$\begin{aligned}\cancel{\Psi'} \mu + \Psi \mu' &= \cancel{A \Psi \mu} + g(t) \Rightarrow \Psi \mu' = g(t) \Rightarrow \mu' = \Psi^{-1}(t) g(t) \\ &= \frac{1}{2} \underbrace{\begin{bmatrix} 3e^{-t} & e^{-t} \\ -e^{-t} & e^t \end{bmatrix}}_{\Psi^{-1}(t)} \underbrace{\begin{bmatrix} e^t \\ t \end{bmatrix}}_{g(t)}\end{aligned}$$

so, $\vec{\mu}' = \frac{1}{2} \begin{bmatrix} (3-t)e^{-t} \\ -e^{2t} + te^t \end{bmatrix}$ integrate it now

$$\vec{\mu} = \frac{1}{2} \begin{bmatrix} \int (3-t)e^{-t} dt \\ \int (-e^{2t} + te^t) dt \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3t - (-te^{-t} - e^{-t}) + d_1 \\ -\frac{e^{2t}}{2} + te^t - e^t + d_2 \end{bmatrix} \begin{matrix} = 0 \text{ wlog} \\ 0 \text{ wlog} \end{matrix}$$

$$\int te^t dt = -te^{-t} + \int e^{-t} dt = -te^{-t} - e^{-t} \quad (\text{IBP})$$

$$\vec{x}_p = \Psi(t) \vec{\mu}(t)$$

$$\vec{x}_p = \frac{1}{2} \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} 3t + te^{-t} + e^{-t} \\ -\frac{e^{2t}}{2} + te^t - e^t \end{bmatrix} = ? \text{ exercise}$$

$$\vec{x}_g = \vec{x}_h + \vec{x}_p$$

$$\vec{x}_g = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \vec{c} + \vec{x}_p$$

ex solve $\vec{x}' = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_A \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{g(t)}$

x_h

eigenvalues: $(1-\lambda)^2 = 0 \quad \lambda_1 = \lambda_2 = 1 \quad \text{alg}(\lambda) = 2$

eigenvectors: $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} v_1 \text{ free} \rightarrow m(\lambda) = 1 < \text{alg}(\lambda) = 2 \\ v_2 \text{ leading} \Rightarrow \text{generalized eigenvector} \end{matrix}$

$\ker(A-I) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\} : v_2 = a_1 v_1, \text{free} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\ker(A-I)^2 = (A-I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

now choose $\vec{v}^{(2)} \in (A-I)^2$
 $\vec{v}^{(2)} \notin (A-I)$

wlog choose $\vec{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

find $\vec{v}^{(1)} = (A-I) \vec{v}^{(2)}$

$\vec{v}^{(1)} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in \ker(A-I)$

$\vec{j} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad e^{\vec{j}t} = e^t \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \vec{j} = P^{-1}AP \rightarrow A = P\vec{j}P^{-1}$
 $\mathcal{O}_{\text{alg}(\lambda)}^{2 \times 2}$

$\vec{x}_h = \underbrace{e^{At} P}_{\vec{\Psi}(t)} \cdot \vec{c} = P e^{\vec{j}t} \underbrace{P^{-1} P}_{I} \vec{c} = \underbrace{P e^{\vec{j}t}}_{\vec{\Psi}(t)} \vec{c}$

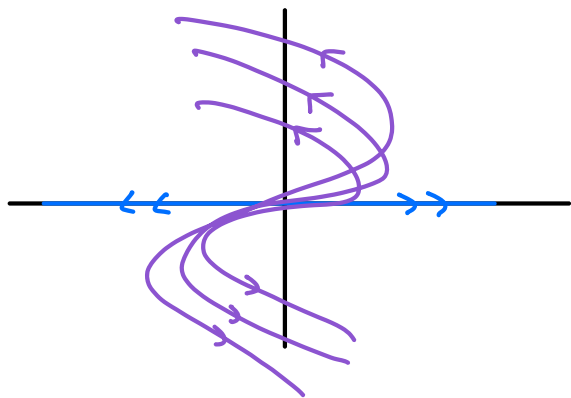
where $P = (\vec{v}^{(1)} \vec{v}^{(2)}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\vec{x}_h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \vec{c} = \begin{pmatrix} -e^t & -te^t \\ 0 & e^t \end{pmatrix} \vec{c}$

$\vec{x}_p = \vec{\Psi}(t) \vec{\mu}(t)$ subs in $\vec{x}' = A\vec{x} + \vec{g} \Rightarrow \vec{\Psi}(t) \vec{\mu}'(t) = \vec{g}(t) \Rightarrow \mu(t) = \vec{\Psi}^{-1}(t) \vec{g}(t)$

$\hookrightarrow \vec{\Psi}(t) \vec{\mu}' = \vec{g}(t)$

$\begin{bmatrix} -e^t & -te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} \mu_1' \\ \mu_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ solve by Cramer's rule



OR CRAMER'S RULE

CRAMER'S RULE

Let A be a square matrix and $A\vec{x} = \vec{b}$ be a system of linear equations. For any solution $\vec{x}$, we have

$$A\vec{x} = \vec{b} \Rightarrow \text{adj}(A) A\vec{x} = \text{adj}(A) \cdot \vec{b}$$

$$\det(A) \cdot \vec{x} = \text{adj}(A) \cdot \vec{b}$$

$$\det(A) x_k = b_1 a_{1k} + b_2 a_{2k} + \dots + b_n a_{nk} = D_k$$

Note that D_k is the determinant of the matrix obtained from A by replacing the k -th column by $\vec{b}$

ex Solve the system

$$\begin{cases} x+y-z=6 \\ 3x-2y+z=-5 \\ x+3y-2z=14 \end{cases} \Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 14 \end{pmatrix}$$

$$\det(A) = -3$$

$$x = \frac{\begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix}}{-3} = \frac{6 \cdot (4-3) - 1(10-14) - 1(-5+28)}{-3} = \frac{6+4-13}{-3} = -1$$

$$y = \frac{\begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & 2 \end{vmatrix}}{-3} = \frac{1(-10-1) - 6(2-1) - 1(6-5)}{-3} = \frac{-11-6-1}{-3} = 3$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}}{-3} = \frac{1(-28-5) - 6(42-2) - 1(42-2)}{-3} = \frac{-33-114-40}{-3} = -2$$

$$\vec{u}(t) = \begin{bmatrix} 2e^{-t} + te^{-t} \\ -e^{-t} \end{bmatrix}$$

$$\vec{x}_p = \Psi(t) \vec{u}(t) = \begin{pmatrix} -e^t & -te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 2e^{-t} + te^{-t} \\ -e^{-t} \end{pmatrix} = \begin{bmatrix} -2-t+t \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\vec{x}_g = \vec{x}_h + \vec{x}_p \Rightarrow \begin{bmatrix} -e^t & -te^t \\ 0 & e^t \end{bmatrix} \vec{c} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\text{ex } \vec{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$\vec{x}_h:$$

$$\det(A - \lambda I) = \lambda^2 + 1 = 0 \quad \lambda = \pm i \quad \text{Re}(\lambda) = 0 \quad \text{Im}(\lambda) = 1$$

$$\lambda = i \quad \det(A - iI)$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2-i \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 - (2+i)v_2 = 0 \quad \vec{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\mu_1' = \frac{\begin{pmatrix} 1 & -te^t \\ 1 & e^t \end{pmatrix}}{-e^{2t}} = \frac{e^t + te^t}{-e^{2t}}$$

$$\mu_1' = -e^{-t} - te^{-t}$$

$$\mu_1 = \int (-e^{-t} - te^{-t}) dt = e^{-t} + te^{-t} + e^{-t} + d_1 = 0$$

$$\mu_2' = \frac{\begin{vmatrix} -e^t & 1 \\ 0 & 1 \end{vmatrix}}{-e^{2t}} = + \frac{e^t}{e^{2t}} = e^{-t}$$

$$\mu_2 = -e^{-t} + d_2 = 0$$

$$\vec{x}_1 = e^{\lambda t} \vec{v} = e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos t + i \sin t) \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\Rightarrow \underbrace{\left(\cos t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}_{\text{Re}(\vec{x}_1) = \vec{x}^{(1)}} + i \underbrace{\left\{ \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}}_{\text{Im}(\vec{x}_1) = \vec{x}^{(2)}}$$

$$x_n = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)}$$

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + 2 \sin t \\ \sin t \end{bmatrix}$$

$$\vec{x}_p = y(t) \vec{u}' = g(t) \quad \det y(t) = -1$$

$$\mu_1' = \frac{\begin{vmatrix} 1 & \cos t + 2 \sin t \\ t & \sin t \end{vmatrix}}{-1} = -(\sin t - t \cos t - 2t \sin t) \quad [\text{integrate}]$$

$$\mu_1 = \int (-\sin t + t \cos t + 2t \sin t) dt = 2 \cos t + 2 \sin t + t \sin t - 2t \cos t + d_1 = 0$$

$$\mu_2' = \frac{\begin{vmatrix} 2 \cos t - \sin t & 1 \\ \cos t & t \end{vmatrix}}{-1} \quad [\text{integrate}]$$

$$\mu_2(t) = \underbrace{(2-2t) \sin t - (2+t) \cos t}_{\mu_2} + d_2 = 0 \quad \text{so, } \vec{u} = \begin{bmatrix} \bar{\mu} \\ \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} d \\ d_1 \\ d_2 \end{bmatrix}$$

Higher Order Linear ODEs

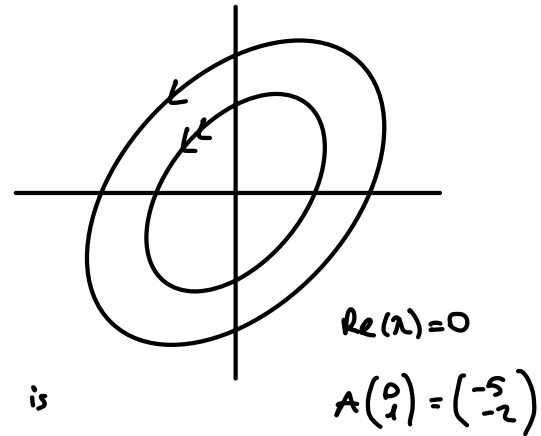
$$p_n(t) y^{(n)} + p_{n-1}(t) y^{(n-1)} + \dots + p_1(t) y' + p_0(t) y = G(t) \rightarrow n\text{th order linear ODE if } p_n(t) \neq 0$$

if $p_n(t) \neq 0 \quad \forall t \in I$ then divide DE with $p_n(t)$

$$y^{(n)} + \frac{p_{n-1}(t)}{p_n(t)} y^{(n-1)} + \dots + \frac{p_1(t)}{p_n(t)} y' + \frac{p_0(t)}{p_n(t)} y = \frac{G(t)}{p_n(t)} \quad \text{OR}$$

$$y^{(n)} + a_{n-1}(t) y^{(n-1)} + \dots + a_1(t) y' + a_0(t) y = g(t) \quad \text{"Standard form of } n\text{th order linear ODE"}$$

$$\text{with } a_i = \frac{p_i(t)}{p_n(t)} \quad \text{and } g(t) = \frac{G(t)}{p_n(t)}$$



Defn:

if $g(t) = 0 \Rightarrow$ homogeneous (H)

if $g(t) \neq 0 \Rightarrow$ non homogeneous (N)

notation if $a_i(t)$ $i=0,1,\dots,n-1$ are cont. fn. on interval I then we say the

differential operator $\mathcal{L}$ by the eqn ($\mathcal{L}$ because linear operator)

$$\mathcal{L}[y] = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y \quad \text{where the operator}$$

$$\mathcal{L} = D^n + a_{n-1}(t)D^{n-1} + \dots + a_1(t)D + a_0(t) \quad \text{with} \quad D^n = \frac{d^n}{dt^n}$$

ex $\mathcal{L} = D^2 + D - 1 \Rightarrow \mathcal{L}[y] = y'' + y' - y$

$$= \frac{d^2}{dt^2} + \frac{d}{dt} - 1$$

$$\mathcal{L}[y] = y'' + y' - y, \quad D^n = \frac{d^n}{dt^n}$$

$$\underline{y''' - 2ty'' + y = 0}$$

(N) $\mathcal{L}[y] = g(t)$

where $\mathcal{L} = D^n + a_{n-1}(t)D^{n-1} + \dots + a_1(t)D + a_0(t)$

(H) $\mathcal{L}[y] = 0$

Remark $\mathcal{L}$ is chosen to remind us operator $\mathcal{L}$ is linear:

$$\mathcal{L}[ay_1 + by_2] = a\mathcal{L}[y_1] + b\mathcal{L}[y_2]$$

ex is $\mathcal{L} = D^2 + D - 1$ linear?

$$\mathcal{L}[\alpha y_1 + \beta y_2] = D^2(\alpha y_1 + \beta y_2) + D(\alpha y_1 + \beta y_2) - 1(\alpha y_1 + \beta y_2)$$

$$= \alpha y_1'' + \alpha y_1' - \alpha y_1 + \beta y_2'' + \beta y_2' - \beta y_2$$

$$= \alpha(y_1'' + y_1' - y_1) + \beta(y_2'' + y_2' - y_2) \quad \square$$

ex consider 2nd linear ODE

$$y'' + a_1(t)y' + a_0(t)y = g(t) \Rightarrow \mathcal{L}[y] = g(t) \quad \mathcal{L} = D^2 + a_1(t)D + a_0(t)$$

ex $y''' - 2y'' + y' - y = e^{-t} \Rightarrow \mathcal{L}[y] = e^{-t}$ where

$$\mathcal{L} = D^3 - 2D^2 + D - 1$$

Defn define $\mathcal{L}[y] = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y$ where $a_i(t)$ cont on int. I

① (N) $\mathcal{L}[y] = g(t)$, $y(t)$ is cont in I

(H) $\mathcal{L}[y] = 0$

Now give IC:

② $y(t_0) = y_0$, $y'(t_0) = y'_0$, $y''(t_0) = y''_0$... $y^{(n-1)}(t_0) = y^{(n-1)}_0$

where $y_0, y'_0, y''_0, \dots, y^{(n-1)}_0$ are some given values

then ① + ② is called IVP which is n -th order linear.

Thm consider IVP $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = g(t)$

$$y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y^{(n-1)}_0$$

where $a_i(t)$, $i=0, \dots, n-1$ and $g(t)$ are cont on open interval I containing t_0

then $\exists$ exactly 1 sol $y = \phi(t)$ of the IVP satisfying both DE and IC

ex $3y''' + x^2y'' + 4xy' + 8y = e^x$, $y(1) = 3$, $y'(1) = 6$, $y''(1) = -3$ (linear 3rd degree)

$\Downarrow$

you have to leave the biggest y' alone!

$$\underbrace{y'''}_{a_2} + \underbrace{\frac{x^2}{3}y''}_{a_1} + \underbrace{\frac{4x}{3}y'}_{a_0} + \underbrace{\frac{8}{3}y}_{g(t)} = \frac{e^x}{3}$$

a_i and $g(t)$ all cont. interval of sol $\mathbb{R}$ by EU

ex determine the intervals for which the unique sol exists for IC

a) $x(x-1)y'' + 3xy' + 4y = 2$ (2nd order linear)

$$y'' + \underbrace{\frac{3x}{x(x-1)}}_{a_1}y' + \underbrace{\frac{4}{x(x-1)}}_{a_0}y = \underbrace{\frac{2}{x(x-1)}}_{g(t)}$$

$$\begin{array}{c} \text{I} \quad \quad \text{II} \quad \quad \text{III} \\ \hline (-\infty, 0) \quad 0 \quad (0, 1) \quad 1 \quad (1, \infty) \end{array}$$

$a_1, a_0, g(x)$ cont $\forall x \in \mathbb{R} - \{0, 1\}$

by EU: if $x_0 \in (-\infty, 0)$ IVP has unique sol on I

if $x_0 \in (0, 1)$ IVP has unique sol on II

if $x_0 \in (1, \infty)$ IVP has unique sol on III

b) $e^x y'' + x y' + y = \tan x$

$$\Downarrow$$

$$y'' + \underbrace{\frac{x}{e^x}}_{a_1} y' + \underbrace{\frac{1}{e^x}}_{a_0} y = \underbrace{\frac{\tan x}{e^x}}_{g(x)}$$

a_1, a_0 cont $\forall x \in \mathbb{R}$ however $g(x) = \frac{\sin x}{e^x \cos x}$ is cont $\forall x$ where $\cos x \neq 0$
 $x = (2k+1)\frac{\pi}{2}$
 $k = 0, \pm 1, \pm 2, \dots$

$$\begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ \\ -\frac{5\pi}{2} & -\frac{3\pi}{2} & -\frac{\pi}{2} & \frac{\pi}{2} & \frac{3\pi}{2} \end{array}$$

$x_0 = 2\pi \rightarrow I = (\frac{3\pi}{2}, \frac{5\pi}{2})$

$x_0 = 3\pi \rightarrow I = (\frac{5\pi}{2}, \frac{7\pi}{2})$

if $x \in ((2k+1)\frac{\pi}{2}, (2k+3)\frac{\pi}{2})$ the soln exist by EU

Thm (Superposition Principle) if the fns $y_1, y_2, \dots, y_n$ are solns of

$$L[y] = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = 0 \quad (H)$$

then any linear combination of these solns

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \quad c_i: \text{constant} \quad \text{is a soln for } (H)$$

Pf if $y_1, y_2, \dots, y_n$ are soln of $L[y] = 0 \Rightarrow L[y_1] = 0, \dots, L[y_n] = 0$

$$L[c_1 y_1 + c_2 y_2 + \dots + c_n y_n] = c_1 L[y_1] + c_2 L[y_2] + \dots + c_n L[y_n] = 0 \quad \square$$

$\downarrow$
by linearity

Q: is there any soln which can NOT be obtained from $y = c_1 y_1 + \dots + c_n y_n$

where $y_1, y_2, \dots, y_n$ are solns of $L[y] = 0 \quad (H)$

A: NO! if $y_1, \dots, y_n$ are linearly independent, in other words, if $\exists$ point to where the

Wronskian: $W(y_1, y_2, \dots, y_n)(t_0) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$

then the fns $y_1, y_2, \dots, y_n$ are linearly independent

Defn: Then the family of solns $y = c_1 y_1 + \dots + c_n y_n$ with random c_i includes every soln to (H)

Thus it's called a **general soln** and the lin. indep. set $\{y_1, \dots, y_n\}$ is called

fundamental set of solns of (H)

Thm Let $\{y_1(t), y_2(t) \dots y_n(t)\}$ be a set of solns for

$$(H) \quad \mathcal{L}[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = 0. \quad \text{Then followings are equivalent:}$$

$$1) \{y_1, \dots, y_n\} \text{ is linearly indep.} \Rightarrow c_1 y_1 + \dots + c_n y_n = 0 \Rightarrow c_i = 0 \quad \forall i$$

$$2) W(y_1, y_2, \dots, y_n)(t) \neq 0 \quad \forall t \in I$$

$$3) W(y_1, \dots, y_n)(t_0) \neq 0 \quad \text{for some } t_0 \in I$$

$$4) y = c_1 y_1 + \dots + c_n y_n \text{ is general soln for } (H)$$

$$5) \text{ for NON-HOMOGENOUS system } \mathcal{L}[y] = g(t) \quad (N)$$

$$y = \underbrace{y_h}_{\text{homog complementary soln for (H)}} + \underbrace{y_p}_{\text{particular soln}} \text{ is a general soln of (N)}$$

$$6) \text{ If } \Phi_1(t) \text{ and } \Phi_2(t) \text{ are solns for (N) then } y = \Phi_1(t) - \Phi_2(t) \text{ is a soln for (H)}$$

Pf 6) since $\Phi_1(t)$ and $\Phi_2(t)$ are solns of (N) then

$$\begin{aligned} \mathcal{L}(\Phi_1) &= g(t) \\ \mathcal{L}(\Phi_2) &= g(t) \end{aligned} \Rightarrow \mathcal{L}(\Phi_1 - \Phi_2) \underset{\substack{\downarrow \\ \text{by linearity of } \mathcal{L}}}{=} \mathcal{L}(\Phi_1) - \mathcal{L}(\Phi_2) = 0$$

$$\Rightarrow \mathcal{L}[\Phi_1 - \Phi_2] = 0 \Rightarrow \Phi_1 - \Phi_2 \text{ is a soln for } \mathcal{L}[y] = 0 \quad (H)$$

Solution techniques for 1-th order linear DE:

$$(N) \quad y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = g(t)$$

$$(H) \quad y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = 0$$

Now consider first non-hom and hom DE with constant coefficients, $a_i(t) = a_i$ constants $\forall i: 0 \leq i \leq n-1$

$$(\tilde{N}) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t)$$

$$(\tilde{H}) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

first consider $(\tilde{H})$

(H) can be converted into a system of 1st order ODE:

that is:

$$\begin{array}{lcl}
 x_1 = y & & x_1' = y' = x_2 \\
 x_2 = y' & \xrightarrow{\text{Differentiate}} & x_2' = y'' = x_3 \\
 x_3 = y'' & & x_3' = y''' = x_4 \\
 \vdots & & \vdots \\
 x_n = y^{(n-1)} & & x_n' = y^{(n)} = -a_{n-1}y^{(n-1)} - a_{n-2}y^{(n-2)} - \dots - a_1y' - a_0y
 \end{array}$$

$\underbrace{\hspace{10em}}_{\text{matrix } A}$

$$x_n' = -a_{n-1}x_n - a_{n-2}x_{n-1} - \dots - a_1x_2 - a_0x_1$$

then

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

we know how to solve this system of 1st order lin. eqns. with constant coeff. matrix A.

so look for characteristic eqn $\lambda_1, \lambda_2, \dots, \lambda_n$

$\det(A - \lambda I) = 0$ to find eigenvalues. Then obtain corresponding eigenvectors $\vec{v}^{(1)}, \dots, \vec{v}^{(n)}$

so the soln: $\vec{x}^{(1)}: e^{\lambda_1 t} \vec{v}^{(1)}, \vec{x}^{(2)}: e^{\lambda_2 t} \vec{v}^{(2)}, \dots, \vec{x}^{(n)} = e^{\lambda_n t} \vec{v}^{(n)}$

now calculate $\det(A - \lambda I)$:

$$\begin{vmatrix} -\lambda & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & -\lambda \\ -a_0 & -a_1 & \dots & \dots & -\lambda - a_{n-1} \end{vmatrix} \Rightarrow \det(A - \lambda I) = (-1)^n (\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0) = 0$$

$$\Rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \quad \text{"char. eqn"}$$

Remark: since $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0$, it is observed that one can obtain the char eqn by putting λ instead of y in (H).

ex $y'' + a_1 y' + a_0 y = 0$

$$\begin{array}{l} x_1 = y \\ x_2 = y' \end{array} \xrightarrow{\text{DIFF}} \begin{array}{l} x_1' = y' = x_2 \\ x_2' = y'' = -a_1 y' - a_0 y = -a_1 x_2 - a_0 x_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -a_0 & -\lambda - a_1 \end{vmatrix} = \lambda^2 + a_1 \lambda + a_0 = (-1)^2 (\lambda^2 + a_1 \lambda + a_0) = 0$$

$$= \boxed{\lambda^2 + a_1 \lambda + a_0 = 0}$$

ex $y''' + a_2 y'' + a_1 y' + a_0 y = 0$

$$\begin{array}{l} x_1 = y \\ x_2 = y' \\ x_3 = y'' \end{array} \xrightarrow{\text{DIFF}} \begin{array}{l} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = -a_2 y'' - a_1 y' - a_0 y = -a_2 x_3 - a_1 x_2 - a_0 x_1 \end{array}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -\lambda - a_2 \end{vmatrix} = (-1)^3 \cdot (\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0) = 0$$

Remark: Once we find the roots of char. eqn. we know how to solve the system.

we have 3 cases for roots of char eqn $\det(A - \lambda I) = 0$

case I: Real Distinct roots for Char Eqn.

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$$\begin{array}{l} x_1 = y \\ x_2 = y' \\ \vdots \\ x_n = y^{(n-1)} \end{array} \longrightarrow \vec{x}' = A \vec{x} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix}$$

$$\text{char. eqn. } \det(A - \lambda I) = \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

if roots are real-distinct then A is diagonalizable, that is $\exists$ an invertible matrix P

$$\text{s.t. } P^{-1} A P = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}, \quad P = [\vec{v}^{(1)}, \vec{v}^{(2)}, \dots, \vec{v}^{(n)}]$$

then the soln of system is

$$\begin{aligned}\vec{x}_{gen} &= \Psi(t) \vec{c} = P e^{Jt} \vec{c} = [\vec{v}^{(1)} \vec{v}^{(2)} \dots \vec{v}^{(n)}] \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= \left[\underbrace{\vec{v}^{(1)} e^{\lambda_1 t}}_{\vec{x}_1} \quad \underbrace{\vec{v}^{(2)} e^{\lambda_2 t}}_{\vec{x}_2} \quad \dots \quad \underbrace{\vec{v}^{(n)} e^{\lambda_n t}}_{\vec{x}_n} \right] \vec{c} \\ &= c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n\end{aligned}$$

in this case:

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix} \quad \text{when} \quad \begin{matrix} x_1 = y \\ x_2 = y' \\ \vdots \\ x_n = y^{(n-1)} \end{matrix} \quad \Rightarrow \quad \begin{matrix} x_{1i} = y_i \\ x_{2i} = y_i' \\ \vdots \\ x_{ni} = y_i^{(n-1)} \end{matrix}$$

$$x_i = \begin{bmatrix} y_i \\ y_i' \\ \vdots \\ y_i^{(n-1)} \end{bmatrix} \quad \text{are the solns of} \quad \vec{x}' = A \vec{x}$$

$$\Psi(t) = [\vec{x}_1 \quad \vec{x}_2 \quad \dots \quad \vec{x}_n] = \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$$

for the soln of 1-th order ODE
look for the entries of first row
of fundamental matrix

Remark we know that

$$\det \Psi(t) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \dots & \vec{x}_n \\ x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} = \overbrace{\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}}^{W(y_1, y_2, \dots, y_n)} \neq 0 \quad \text{from system}$$

so wronskian of $y_1, y_2, \dots, y_n \neq 0 \Rightarrow \{y_1, y_2, \dots, y_n\}$ is a linearly independent

so it constructs a set of fund. soln. for higher order DE $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$ (H)

so general soln $y_{gen} = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$. Since,

$$\Psi(t) = \begin{bmatrix} v_{11} e^{\lambda_1 t} & v_{12} e^{\lambda_2 t} & \dots & v_{1n} e^{\lambda_n t} \\ v_{21} e^{\lambda_1 t} & v_{22} e^{\lambda_2 t} & \dots & v_{2n} e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} e^{\lambda_1 t} & v_{n2} e^{\lambda_2 t} & \dots & v_{nn} e^{\lambda_n t} \end{bmatrix} \Rightarrow \left\{ y_1 = \overbrace{v_{11} e^{\lambda_1 t}}^{\text{constants}}, y_2 = \overbrace{v_{12} e^{\lambda_2 t}}^{\text{constants}}, \dots, y_n = \overbrace{v_{1n} e^{\lambda_n t}}^{\text{constants}} \right\}$$

so, $\mathcal{Y}(t) \Rightarrow \{y_1 = e^{\lambda_1 t}, y_2 = e^{\lambda_2 t}, \dots, y_n = e^{\lambda_n t}\}$ is the fundamental set of solns.

ex $y'' - 2y' - 3y = 0$

1st way: (use system)

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned} \xrightarrow{\text{diff}} \begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = 2x_2 + 3x_1 \end{aligned} \Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}^A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \rightarrow (\lambda - 3)(\lambda + 1) = 0 \quad \lambda = 3, \lambda = -1 \text{ real distinct}$$

$$\lambda = 3: \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow 3v_1 - v_2 = 0 \Rightarrow 3v_1 = v_2 \text{ so, } V_{\lambda=3} = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \Rightarrow x_1(t) = e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda = -1: \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 \text{ so } V_{\lambda=-1} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \Rightarrow x_2(t) = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so, $\mathcal{Y}(t) = [x_1 \ x_2] = \begin{bmatrix} \overset{y_1}{\circledast} e^{3t} & \overset{y_2}{\circledast} -e^{-t} \\ 3e^{3t} & e^{-t} \end{bmatrix}$ so, $y_{gen} = c_1 e^{3t} - c_2 e^{-t} = c_1 e^{3t} + c_3 e^{-t}$

2nd way: (write char eqn)

$$y'' - 2y' - 3y = 0$$

$\Downarrow$

$$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0$$

so $\lambda = 3$ and $\lambda = -1$ real distinct

$$y_1 = e^{3t} \quad y_2 = e^{-t}$$

so,

$$y_g = c_1 e^{3t} + c_2 e^{-t}$$

Case II (Complex roots for Char. Eqn)

If $\lambda_1 = \alpha + i\beta$ is a complex root when $\lambda_2 = \bar{\lambda}_1 = \alpha - i\beta$ is also a root.

and λ_1 and λ_2 produce sols of the form

$$\{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t\}$$

Now by case I

the sols are since $\lambda_1 \neq \lambda_2$ (distinct)

$$\phi_1 = e^{\lambda_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$$

$$\phi_2 = e^{\lambda_2 t} = e^{(\alpha - i\beta)t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t} (\cos \beta t - i \sin \beta t)$$

but we look for real sols:

$$y_1 = \frac{\phi_1 + \phi_2}{2} = e^{\alpha t} \cdot \cos \beta t = \operatorname{Re}(\phi_1), \quad y_2 = \frac{\phi_1 - \phi_2}{2} = e^{\alpha t} \cdot \sin \beta t = \operatorname{Im}(\phi_1)$$

$$y_1 = \operatorname{Re}(\phi_1) = e^{(\operatorname{Re} \lambda)t} \cdot \cos(\operatorname{Im} \lambda)t$$

$$y_2 = \operatorname{Im}(\phi_1) = e^{(\operatorname{Re} \lambda)t} \cdot \sin(\operatorname{Im} \lambda)t$$

ex $y'' + 4y = 0$ solve

1st way: $\lambda^2 + 4\lambda = 0 \quad \lambda = \mp 2i \quad \Rightarrow \quad \begin{aligned} y_1 &= e^{0t} \cos 2t \\ y_2 &= e^{0t} \sin 2t \end{aligned}$

$$y = c_1 y_1 + c_2 y_2$$

2nd way (using systems):

$$x_1 = y \quad x_1' = y' = x_2 \quad \Rightarrow \quad x_2' = y'' = -4y = -4x_1$$

$$x_2 = y' \quad \Rightarrow \quad \underbrace{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}_A = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{char eqn: } \det(A - \lambda I) = \lambda^2 + 4 = 0 \Rightarrow \lambda = \mp 2i$$

$$\lambda = 2i: \quad \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \rightarrow \begin{bmatrix} -2i & 1 \\ 0 & 0 \end{bmatrix}$$

$$(\text{char}(A - \lambda I)) \quad -2i v_1 + v_2 = 0 \quad \begin{aligned} v_1 &= 1 \\ v_2 &= 2i \end{aligned}$$

$$v = \begin{bmatrix} 1 \\ 2i \end{bmatrix} \rightarrow \vec{x}_1 = e^{\lambda t} \vec{v} = e^{2it} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = (\cos 2t + i \sin 2t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right]$$

$$= \underbrace{(\cos 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 2 \end{pmatrix})}_{\operatorname{Re}(x_1)} + i \underbrace{(\cos 2t \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix})}_{\operatorname{Im}(x_1)}$$

$$y(t) = [x_1, x_2] = \begin{bmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{bmatrix} \quad y = c_1 y_1 + c_2 y_2$$

ex $y'' + y' + y = 0$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\lambda_{1,2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_1 = e^{-1/2t} \cdot \cos(\sqrt{3}/2t), \quad y_2 = e^{-1/2t} \sin(\sqrt{3}/2t)$$

$$y_g = c_1 y_1 + c_2 y_2$$

2nd way using system exercise

ex $y''' + y' = 0$

$$\lambda^3 + \lambda' = 0$$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda = 0$$

$$\downarrow$$

$$e^{0t}$$

$$y_1 = 1$$

$$\lambda = \pm i$$

$$\downarrow$$

$$e^{0t} \cos t$$

$$y_2 = \cos t$$

$$\downarrow$$

$$e^{0t} \sin t$$

$$y_3 = \sin t$$

$$y_g = c_1 \cdot 1 + c_2 \cos t + c_3 \sin t$$

case III (Repeated roots for char eqns)

Thm let λ be a root for char eqn

$$\det(A - \lambda I) = (-1)^n (\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0) = 0 \quad [\text{obtained from higher order} \dots]$$

with $\text{alg}(\lambda) = k$ then there has to be ONLY ONE Jordan Block

ie $J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \lambda \end{bmatrix}_{k \times k}$ for the soln of the system corresponding repeated eigenvalue λ

where the system is

$$\vec{x}' = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \vec{x}$$

$$e^{J_\lambda t} = e^{\lambda t} \begin{pmatrix} 1 + t^2/2 & \dots & t^{k-1}/(k-1)! \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}_{k \times k}$$

consider
only the
first row

Then the first row of $\Psi(t)$

$$\underbrace{[v_{11} e^{\lambda t}]}_{\bar{y}_1} \quad \underbrace{[v_{11}t + v_{12}] e^{\lambda t}}_{\bar{y}_2} \quad \underbrace{[v_{11} \frac{t^2}{2} + v_{12}t + v_{13}] e^{\lambda t}}_{\bar{y}_3} \quad (v_{11} \frac{t^3}{3!} + v_{12} \frac{t^2}{2!} + v_{13}t + v_{14}) e^{\lambda t} \dots \underbrace{[v_{11} \frac{t^{k-1}}{(k-1)!} + v_{12} \frac{t^{k-2}}{(k-2)!} + \dots + v_{1k}] e^{\lambda t}}_{\bar{y}_k}$$

$y = d_1 \bar{y}_1 + d_2 \bar{y}_2$ is also a soln

$$y = d_1 v_{11} e^{\lambda t} + d_2 v_{11} t e^{\lambda t} + d_2 v_{12} e^{\lambda t}$$

which gives n lin. indep. solns st

happens bc of
only 1 Jordan blk
↓

$$y = \underbrace{(d_1 v_{11} + d_2 v_{12})}_{c_1} e^{\lambda t} + \underbrace{d_2 v_{11} t}_{c_2} e^{\lambda t}$$

$$\{y_1 = e^{\lambda t}, y_2 = t e^{\lambda t}, y_3 = t^2 e^{\lambda t}, y_k = t^{k-1} e^{\lambda t}\}$$

which is a fun. set of solns for

$$(H) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Remark: consider 2 block J

$$J = \left(\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{array} \right) \longrightarrow e^{Jt} = \left(\begin{array}{c|cc} e^{\lambda t} & 0 & 0 \\ \hline 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{array} \right)$$

$$\Psi(t) = \underbrace{\begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix}}_P \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & t e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix} = \begin{bmatrix} v_{11} e^{\lambda t} & v_{12} e^{\lambda t} & v_{12} t e^{\lambda t} + v_{13} e^{\lambda t} \end{bmatrix}$$

brings us $\{e^{\lambda t}, e^{\lambda t}, t e^{\lambda t}\}$ only two linearly indep. solns. but we need three

this case never happens for a system obtained from a higher order ODE by this.

ex $y'' + 6y' + 9y = 0$, $y(0)=1$, $y'(0)=5$

1st way: $\lambda^2 + 6\lambda + 9 = 0$ $(\lambda+3)^2 = 0$ $\lambda = -3$ $\text{alg}(\lambda) = 2$

$$y_1 = e^{-3t} \quad y_2 = t e^{-3t} \quad y_3 = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$y(0)=1 = c_1 \quad y'(0)=5 \rightarrow y'_3 = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t} \Big|_{t=0}$$

$$5 = -3c_1 + c_2 \Rightarrow c_2 = 8$$

$$y(t) = e^{-3t} + 8t e^{-3t}$$

2nd way of solving $y'' + \underset{a_1}{6}y' + \underset{a_0}{9}y = 0 \quad y(0) = 1 \quad y'(0) = 5$

$$\vec{x}' = A \vec{x} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \vec{x} \quad \det(A - \lambda I) = \lambda^2 + 6\lambda + 9 = 0 \quad \lambda = -3 \quad \mu(\lambda) = 2$$

$\lambda = -3 : \ker(A + 3I)$

$$\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \quad 3v_1 + v_2 = 0 \quad \text{span} \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$$

$$\ker(A + 3I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \ker(A + 3I)^2 \quad \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

choose $\vec{v}^{(2)} \in \ker(A + 3I)^2$ wlog $\vec{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \vec{v}^{(1)} = (A + 3I)\vec{v}^{(2)} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$
 $\vec{v}^{(1)} \notin \ker(A + 3I)$

$$P = \begin{pmatrix} 3 & 1 \\ -9 & 0 \end{pmatrix} \quad J = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \quad e^{Jt} = e^{-3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\Psi(t) = P e^{Jt} = \begin{bmatrix} 3 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} e^{-3t} & t e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} = \begin{bmatrix} 3e^{-3t} & 3t e^{-3t} + e^{-3t} \\ -9e^{-3t} & -9t e^{-3t} \end{bmatrix}$$

choose $y_1 = e^{-3t} \quad y_2 = t e^{-3t} \quad y y = c_1 y_1 + c_2 y_2$

Remark: if $\alpha \mp i\beta$ is repeated k times

then the real solns

$$\left\{ e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, t e^{\alpha t} \cos \beta t, t e^{\alpha t} \sin \beta t, \dots, t^{k-1} e^{\alpha t} \cos \beta t, t^{k-1} e^{\alpha t} \sin \beta t \right\} \quad 2k \text{ solutions}$$

ex $y^{(4)} + 2y'' + y = 0 \rightarrow \lambda^4 + 2\lambda^2 + 1 = 0 \rightarrow \lambda^2(\lambda^2 + 1) + (\lambda^2 + 1) = 0$

alg $(\lambda = i) = 2 = k \quad \alpha = 0 \quad \beta = 1$
 $(\lambda^2 + 1)(\lambda^2 + 1) = 0$
 $\lambda = \mp i \quad \lambda = \mp i$

$$\{\cos t, \sin t, t \cos t, t \sin t\} \quad y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

ex $y^{(6)} - y'' = 0 \rightarrow \lambda^6 - \lambda^2 = 0$

$$y(t) = \underbrace{c_1 \cdot 1 + c_2 t}_{0^k} + c_3 \cos t + c_4 \sin t + c_5 e^t + c_6 e^{-t}$$

$$\begin{array}{l} \lambda^2(\lambda^4 - 1) = 0 \\ \lambda^2(\lambda^2 - 1)(\lambda^2 + 1) = 0 \\ \downarrow \\ \begin{array}{ccc} e^{0t} & t e^{0t} & \lambda = \mp i \\ \parallel & & \downarrow \\ 1 & & \cos t \quad \sin t \\ \lambda = 0^* & & \end{array} \end{array}$$

Solving Nonhomogeneous Linear Higher Order ODE: (29 Nisan 12)

$$(N) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1(t) y' + a_0(t) y = g(t) \Rightarrow \mathcal{L}[y] = g(t) \quad (N)$$

$$(H) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1(t) y' + a_0(t) y = 0 \Rightarrow \mathcal{L}[y] = 0 \quad (H)$$

$\rightarrow y_{\text{gen}} = y_h + y_p$ and we know how to solve (H) when a_i 's are constant.

$$\text{and } \mathcal{L}(y) = D^n + a_{n-1}(t) D^{n-1} + \dots + a_1(t) D + a_0(t), \quad D^n = \frac{d^n}{dt^n}$$

for y_p (N), we'll see 3 techniques:

① undetermined coefficient method (UCM) : works for some special $g(t)$ and

for DE with constant coefficient ($\tilde{N}$)

② reduction of order

③ variation of parameter (VOP)

} works for variable coefficient DEs (N) and for any $g(t)$. However, they require the soln of corresponding homogs DE, $\mathcal{L}[y] = 0$ (H)

① undetermined coefficient method:
(by using operator method)

UCM is applicable when the coefficients of (N) $\mathcal{L}[y] = g(t)$ are constants and $g(t)$ is in a special form st $g(t) \in \text{Class: } \{ \exp, \cos, \sin, \text{polynomials and their sums and products} \}$

$$\text{ex } g(t): e^{3t}, e^{-t} \sin t, \cos t, 1+t+t^3, 1+t+e^{-t} \cos 2t$$

consider ($\tilde{N}$) $\mathcal{L}[y] = g(t)$ a_i : constant, $\mathcal{L}(y) = D^n + a_{n-1}(t) D^{n-1} + \dots + a_1 D + a_0$

($\tilde{H}$) $\mathcal{L}[y] = 0$ "corresponding hom DE with a_i : constant

$$\begin{array}{ccc} y_g = y_h + y_p & & \\ \downarrow & & \searrow \\ \text{soln to (H)} & & \text{particular soln} \\ \text{we know how} & & \text{to (N)} \\ \text{to find} & & \end{array}$$

to find " y_p " by operator (UCM) method:

write DE in operator form

$$(\bar{N}) \quad \mathcal{L}[y] = \mathcal{L}(D)y = g(t)$$

$$(\bar{H}) \quad \mathcal{L}[y] = \mathcal{L}(D)y = 0$$

① solve $\mathcal{L}(D) = 0$ to find y_h

② find $\mathcal{L}_1(D)$ st $\mathcal{L}_1(D)g(t) = 0$ (ie $g(t)$ is a soln for hom DE $\mathcal{L}_1(D)y = 0$)

③ then when the operator $\mathcal{L}_1$ is applied to $(\bar{N}) \quad \mathcal{L}(D)y = g(t)$

$\mathcal{L}_1(D)\mathcal{L}(D)y = 0$ "hom DE", find soln y for this hom DE *

④ identify particular soln $y_p(t)$ by deleting the terms in y which are common in y_h .

then rest will be particular soln y_p .

ex $y'' - 2y' + y = e^t$ solve DE

2nd order, linear, $g(t) = e^t \in \text{Class} \rightarrow$ constant coeff $\rightarrow$ UCM applicable to find y_p .

$$\mathcal{L}(D) = D^2 - 2D + 1 \Rightarrow (D-1)^2$$

$$y_h: \mathcal{L}(D)y = 0 \Rightarrow (D-1)^2 y = 0$$

$$\text{char eqn } p(\lambda) = (\lambda-1)^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 1$$

$$y_p: \mathcal{L}(D)y_p = e^t$$

$$\{e^t, te^t\} \Rightarrow y_h = c_1 e^t + c_2 t e^t$$

① find $\mathcal{L}_1(D)$ st $\mathcal{L}_1(D)g(t) = 0$

$$g(t) = e^t \Rightarrow \lambda = 1 \Rightarrow p(\lambda) = \lambda - 1 = 0 \Rightarrow \mathcal{L}_1(D) = D - 1$$

$$\text{checking part: } (D-1)e^t = \frac{d(e^t)}{dt} = e^t \Rightarrow e^t - e^t = 0 \Rightarrow \mathcal{L}_1(D) = D - 1$$

② find soln of following hom DE: $\mathcal{L}_1(D)\mathcal{L}(D)y = \mathcal{L}_1(D)g(t) = 0$

$$\Rightarrow (D-1)(D-1)^2 y = 0 \Rightarrow (D-1)^3 y = 0 \Rightarrow \text{char eqn } (\lambda-1)^3 = 0, \lambda = 1^* \quad y \in \{e^t, te^t, t^2 e^t\}$$

$y_p(t) = A t^2 e^t$, A is undetermined coeff, find A by subs $y_p(t)$ in $(\bar{N}) (D-1)^2 y_p = e^t$

$$\Rightarrow \textcircled{1} y_p'' - \textcircled{2} y_p' + \textcircled{1} y_p = e^t, \text{ we know}$$

$$\textcircled{1} y_p = A t^2 e^t$$

$$\textcircled{-2} y_p' = 2A t e^t + A t^2 e^t$$

$$\textcircled{1} y_p'' = 2A e^t + 4A t e^t + A t^2 e^t$$

$$\left. \begin{array}{l} \textcircled{1} y_p = A t^2 e^t \\ \textcircled{-2} y_p' = 2A t e^t + A t^2 e^t \\ \textcircled{1} y_p'' = 2A e^t + 4A t e^t + A t^2 e^t \end{array} \right\} \rightarrow e^t = 2A e^t \Rightarrow A = \frac{1}{2} \Rightarrow y_p = \frac{1}{2} t^2 e^t$$

$$y_g = y_h + y_p \Rightarrow y_g = c_1 e^t + c_2 t e^t + \frac{1}{2} t^2 e^t$$

ex find a linear hom DE with constant coeff of the smallest orders so that

t^2 , $t \cos t$, e^{-t} , $e^t \cos 2t$ are soln.

for t^2 : $t^2 e^{0t} \Rightarrow \lambda = 0^*$, $\underline{\underline{\lambda^3 = 0}}$ if it was $t^3 \rightarrow \lambda^4 = 0$ ok both.

for $t \cos t$: $\lambda = \pm i \Rightarrow \underline{\underline{(\lambda^2 + 1) = 0}}$

for e^{-t} : $\lambda = -1 \Rightarrow \underline{\underline{(\lambda + 1) = 0}}$

for $e^t \cos 2t$: $\lambda = 1 \pm 2i \rightarrow (\lambda - 1) = \pm 2i \Rightarrow (\lambda - 1)^2 = -4 \Rightarrow \underline{\underline{(\lambda - 1)^2 + 4 = 0}}$

so, $p(\lambda) = \underline{\underline{\lambda^3}} \underline{\underline{(\lambda^2 + 1)^2}} \underline{\underline{(\lambda + 1)}} \underline{\underline{((\lambda - 1)^2 + 4)}}$

$$\mathcal{L}(D)y = (D^3(D^2+1)^2(D+1)((D^2-1)^2+4))y = 0$$

ex $y^{(4)} + 2y'' + y = 3 \sin x - 5 \cos x$ (linear, constant coeff $g(x) \in C^\infty$ ucm ✓ for y_p)

$$y_h: \mathcal{L}(D)y = 0 \Rightarrow \mathcal{L}(D) = D^4 + 2D^2 + 1 = (D^2 + 1)^2$$

$p(\lambda) = (\lambda^2 + 1)^2$, $\lambda = \pm i$ repeated 2 times. $\Rightarrow \{\cos t, \sin t, t \cos t, t \sin t\}$

$$y_h = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

$$y_p: \textcircled{1} \mathcal{L}_1(D)g(x) = 0 \quad g(x) = 3 \sin x - 5 \cos x, \quad \lambda = \pm i \rightarrow \lambda^2 = -1 \rightarrow (\lambda^2 + 1) \rightarrow \mathcal{L}_1(D) = D^2 + 1$$

$$\textcircled{2} \mathcal{L}_1(D)g(x) = \mathcal{L}_1(D) \underbrace{\mathcal{L}(D)y}_{g(x)} = 0$$

$$= (D^2 + 1)(D^2 + 1)^2 y = 0 \rightarrow p(\lambda) = (\lambda^2 + 1)^3 = 0 \quad \lambda = \pm i \text{ repeated 3 times}$$

~~$\{\cos t, \sin t, t \cos t, t \sin t\}$~~ , $t^2 \cos t, t^2 \sin t$

$y_p = A t^2 \cos t + B t^2 \sin t$ is the form of y_p where A, B undetermined coeff.

subs y_p in $y^{(4)} + 2y'' + y = 3 \sin x - 5 \cos x$, now find A, B

$$\left. \begin{array}{l} 1/y_p \\ 0/y_p' \\ 2/y_p'' \\ 0/y_p''' \\ 1/y_p^{(4)} \end{array} \right\}$$

exercise

then we get y_p so $y_g = y_h + y_p \quad \square$

ex $y'' - 3y' - 4y = 3e^{2t} + 2\sin t$, $\mathcal{L}(D) = D^2 - 3D - 4 \rightarrow (D-4)(D+1)$ $g(t) \in \mathcal{L} \text{ ops vcmv}$

$y_h: (D-4)(D+1)y = 0$ solve

$p(\lambda) = (\lambda-4)(\lambda+1) = 0$ $\lambda=4$ $\lambda=-1 \Rightarrow \{e^{-t}, e^{4t}\}$ $y_h = c_1 e^{-t} + c_2 e^{4t}$

$y_p: \mathcal{L}_1(0) g(t) = 0$

$g(t) = 3e^{2t} + 2\sin t$
 $\downarrow \quad \downarrow$
 $\lambda=2 \quad \lambda = \alpha \pm i\beta = 0 \pm i = \pm i$

so $p(\lambda) = (\lambda-2)(\lambda^2+1) \Rightarrow \mathcal{L}_1(0) = (D-2)(D^2+1)$

solve $\mathcal{L}_1(0) g(t) = \mathcal{L}_1(0) \mathcal{L}(D) y = 0$

$\Rightarrow (D-2)(D^2+1)(D-4)(D+1)y = 0$

$\{ \underset{\downarrow}{e^{2t}}, \underset{\downarrow}{\cos t}, \cancel{\underset{\downarrow}{e^{4t}}}, \cancel{\underset{\downarrow}{e^{-t}}} \}$ so $y_p = Ae^{2t} + B\cos t + C\sin t$ find A, B, C by subs in $y'' - 4y' = 3e^{2t} + 2\sin t$

$-4/y_p = Ae^{2t} + B\cos t + C\sin t$

$-3/y_p' = 2Ae^{2t} - B\sin t + C\cos t$

$1/y_p'' = 4Ae^{2t} - B\cos t - C\sin t$

+

$3e^{2t} + 2\sin t = \underbrace{-6Ae^{2t}}_3 + \underbrace{\cos t(-4B-3C-B)}_0 + \underbrace{\sin t(-4C+3B+C)}_2$

$A = -1/2$ $\begin{matrix} -5B-3C=0 \\ -5C+3B=2 \end{matrix} \Rightarrow B = 3/17, C = -5/17$ so this means,

$y_p = \frac{-1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t$, and we know $y_h = c_1 e^{-t} + c_2 e^{4t} \Rightarrow y_g = y_h + y_p$

ex find the form of particular soln for $y'' - 4y' = x + 3\cos x + e^{-2x}$

$y_h: p(\lambda) = \lambda^2 - 4\lambda = \lambda(\lambda-4) = 0$

$\lambda=0, \lambda=4 \Rightarrow \{1, e^{2x}, e^{-2x}\}$

$y_p: \mathcal{L}_1(0) g = 0$, $g(x) = x + 3\cos x + e^{-2x}$
 $\downarrow \quad \downarrow \quad \downarrow$
 $\lambda=0^4 \quad \lambda=\pm i \quad \lambda+2$

$p(\lambda) = (\lambda^2) \cdot (\lambda^2+1) (\lambda+2) \Rightarrow \mathcal{L}_1(0) = D^2(D^2+1)(D+2)$

now find soln to

$\mathcal{L}_1(0) \mathcal{L}(D) y = 0$

$$D^2(D^2+1)(D+2)D(D-2)(D+2)y = 0$$

$$\Downarrow$$

$$D^3(D^2+1)(D+2)^2(D-2)y = 0$$

$$\Downarrow$$

$$p(\lambda) = \lambda^3(\lambda^2+1)(\lambda+2)^2(\lambda-2) = 0$$

$$\begin{matrix} \lambda=0^* & \lambda=\pm i & \lambda=-2^* & \lambda=2 \\ (3\text{-times}) & (2\text{-times}) & & \end{matrix}$$

$$\left\{ \cancel{e^{0x}}, \underset{1}{x}e^{0x}, \underset{1}{x^2}e^{0x}, \cos x, \sin x, \cancel{e^{-2x}}, \underset{1}{x}e^{-2x}, \cancel{e^{2x}} \right\} \Rightarrow y_p = (A_1x + A_2x^2)B_1\cos x + B_2\sin x + Cxe^{-2x}$$

exercises find the form of y_p for DE

$$\textcircled{1} D(D-1)y = e^{2x} + \sin x + x+1 \quad (\text{Hint: for } x+1 \quad \downarrow_1 q(x)=0)$$

$$\textcircled{2} y''' + 2y'' - y' - 2y = e^t + t^2$$

$$\textcircled{3} y'' + 2y' + 5y = \underbrace{3xe^{-x}\cos 2x - 2xe^{-2x}\cos x}_{\lambda = -1 \mp 2i \Rightarrow ((\lambda+1)^2 + 4)^2} \quad \lambda = -2 \mp i \Rightarrow ((\lambda+2)^2 + 1)$$

Reduction of Order for y_p requires knowing one soln of (H) DE corresponding to (N)

$$(N) y'' + a_1(t)y' + a_0(t)y = g(t)$$

$$(H) y'' + a_1(t)y' + a_0(t)y = 0$$

Let y_1 be a soln of (H). Assume $y_p = y_1(t)u(t)$, $u(t)$: unknown function

then subs y_p in (N)

$$y_p'' + a_1(t)y_p' + a_0(t)y_p = g(t)$$

$$y_p = y_1 u, \quad y_p' = y_1' u + y_1 u', \quad y_p'' = y_1'' u + 2y_1' u' + u'' \quad \text{subs } * *$$

$$y_1'' u + 2y_1' u' + y_1 u'' + a_1[y_1' u + y_1 u'] + a_0 y_1 u = g(t)$$

$$\underbrace{u(y_1'' + a_1 y_1' + a_0 y_1)}_{0 \rightarrow y_1 \text{ is a soln to (H)}} + u'(2y_1' + a_1 y_1) + u'' y_1 = g(t) \Rightarrow y_1 u'' + (2y_1' + a_1(t)y_1)u' = g(t)$$

$$\text{let } v = u' \rightarrow v' = u''$$

$$\Rightarrow y_1 v' + (2y_1' + a_1(t)y_1)v = g(t)$$

$$(*) \Rightarrow v' + \underbrace{\frac{(2y_1' + a_1(t)y_1)}{y_1}}_P v = \underbrace{\frac{g(t)}{y_1}}_Q \quad \text{FIRST ORDER LINEAR DE}$$

$$\text{so } \mu = e^{\int \frac{2y_1' + a_1(t)y_1}{y_1} dt}$$

if $\mu = e^{\int \frac{2y_1' + a_1(t)y_1}{y_1^2} dt}$ then solve $\oplus$ (first order linear)

then obtain μ by integrating $\int \mu' = \int v \Rightarrow \mu \Rightarrow y_p = \mu y_1$

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ex $y'' - \frac{2}{t^2}y = 1$ solve DE by reduction of order method if $y_1 = t^2$ is a sol to hom DE

$$\boxed{y_p = \mu y_1 = \mu t^2}$$

subs in DE: $y_p' = 2t\mu + t^2\mu'$

$$y_p'' - \frac{2}{t^2}y_p = 1$$

$$y_p'' = 2\mu + \mu t\mu' + t^2\mu''$$

$$\cancel{2\mu} + 4t\mu' + t^2\mu'' - \frac{2}{t^2}\cancel{t^2}\mu = 1 \Rightarrow t^2\mu'' + 4t\mu' = 1$$

$$v = \mu' \Rightarrow v' = \mu'' \Rightarrow t^2v' + 4tv = 1 \Rightarrow v' + \frac{4}{t}v = \frac{1}{t^2} \text{ "1st order linear"}$$

$$\mu(t) = e^{\int \frac{4}{t} dt} = e^{4\ln t} = t^4 \Rightarrow \int (t^4 v)' = \int t^2 \Rightarrow t^4 v = \frac{t^3}{3} + c_1$$

$$v(t) = \frac{1}{3t} + \frac{c_1}{t^4} \Rightarrow \int \mu'(t) = \int \frac{1}{3t} + \frac{c_1}{t^4} \Rightarrow \mu(t) = \frac{1}{3} \ln|t| - \frac{c_1}{3t^3} + c_2$$

$$y_p \text{ was } \mu t^2 \text{ so } y_p = \underbrace{\frac{1}{3} t^2 \ln|t|}_{y_p} - \underbrace{\frac{c_1}{3t}}_{y_h} + c_2 t^2$$

if you keep the constants it gives you homogeneous soln
if you erase it then rest = y_p

so if you keep all constants you end up with general soln

we can say a specific $y_p = \frac{t^2}{3} \ln|t|$

ex $y'' + 2y' + y = \frac{e^{-t}}{t}$ by reduction of order ($\frac{1}{t}$ is not poly. $\frac{1}{t} \notin$ class)

we know constant coeff const, and we can set y_h

$$y_h = \lambda^2 + 2\lambda + 1 = 0 \quad \lambda = -1^* \quad \begin{cases} c_1 e^{-t} + c_2 t e^{-t} = y_h \\ y_1 = e^{-t}, y_2 = t e^{-t} \end{cases}$$

$$y_p = \mu(t) e^{-t} \Rightarrow y_p' = \mu' e^{-t} - \mu e^{-t}$$

$$y_p'' = \mu'' e^{-t} - 2\mu' e^{-t} + \mu e^{-t} \quad (\text{subs in } y_p'' + 2y_p' + y_p = \frac{e^{-t}}{t})$$

$$\Rightarrow y_p(t) = e^{-t}(t \ln|t| + t)$$

$$\Rightarrow e^{-t}\mu'' = \frac{e^{-t}}{t} \Rightarrow \mu'' = \frac{1}{t} \Rightarrow \mu' = \ln|t| + c_1 \quad \text{say } c_1 = 0$$

$$y_g = y_h + y_p$$

$$\mu = t \ln|t| - t + c_2 \quad \text{say } c_2 = 0$$

* keep c_1, c_2 you get
 $y_h = c_1 e^{-t} + c_2 t e^{-t}$

Variation of Parameter: this is applicable for linear non-hom DE with variable

coefficients but we need to know soln of the corresponding hom DE

$$(N) \quad \mathcal{L}[y] = g(t) \quad y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \text{ soln of } (H) \quad \text{then}$$

$$(H) \quad \mathcal{L}[y] = 0 \quad y_p = \mu_1(t) y_1 + \mu_2(t) y_2 + \dots + \mu_n(t) y_n$$

then subs $y_p(t)$ in (N) to find μ_i 's.

first we'll consider

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y_h = c_1 y_1(t) + c_2 y_2(t) \quad [\text{soln for } (H)]$$

$$y_p(t) = \mu_1(t) y_1(t) + \mu_2(t) y_2(t) \quad [\text{soln of } (N)] \quad \text{take derivative}$$

$$y_p'(t) = \boxed{\mu_1' y_1 + \mu_2' y_2} + \mu_1 y_1' + \mu_2 y_2' \quad \text{Main step}$$

force main step to be zero

$$\boxed{y_1 \mu_1' + y_2 \mu_2' = 0} \quad (1)$$

$$y_p''(t) = \mu_1' y_1' + \mu_2' y_2' + \mu_1 y_1'' + \mu_2 y_2''$$

$$\text{subs } y_p''(t), y_p'(t) \text{ and } y_p(t) \text{ in } (N): y_p'' + p(t)y_p' + q(t)y_p = g(t)$$

$$\Rightarrow \mu_1' y_1' + \mu_2' y_2' + \mu_1 y_1'' + \mu_2 y_2'' + p(t)(\mu_1 y_1' + \mu_2 y_2') + q(t)(\mu_1 y_1 + \mu_2 y_2) = g(t)$$

$$\Rightarrow \underbrace{\mu_1 (y_1'' + p(t)y_1' + q(t)y_1)}_{=0} + \underbrace{\mu_2 (y_2'' + p(t)y_2' + q(t)y_2)}_{=0} + \mu_1' y_1' + \mu_2' y_2' = g(t)$$

because y_1, y_2 are solns of (H) lin indep $\Rightarrow \det \neq 0 \Rightarrow$ has a unique soln

$$\Rightarrow \boxed{y_1' \mu_1' + y_2' \mu_2' = g(t)} \quad (2)$$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} \mu_1' \\ \mu_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$y(t) \quad \text{then } W(y_1, y_2) = \det(y(t))$$

$$\mu_1' = \frac{|y_2|}{W(y_1, y_2)} \quad \mu_2' = \frac{|y_1|}{W(y_1, y_2)}$$

Cramers rule

y_j : matrix obtained from y by changing j th column with RHS vector

Eqs of VOP

$$① y_1 \mu_1' + y_2 \mu_2' = 0$$

$$② y_1' \mu_1' + y_2' \mu_2' = g(t)$$

$$\Rightarrow \underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{Y(t)} \begin{bmatrix} \mu_1' \\ \mu_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

CRAMER'S RULE

$$\mu_1' = \frac{|Y_2(t)|}{W(y_1, y_2)}$$

$$\mu_2' = \frac{|Y_1(t)|}{W(y_1, y_2)}$$

note that

$$Y^{-1}(t) = \frac{1}{\det Y} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$$

$$y_p(t) = \mu_1 y_1 + \mu_2 y_2$$

$$y_p(t) = y_1 \int \frac{-y_2 g(t)}{W} dt + y_2 \int \frac{y_1 g(t)}{W} dt$$

if you keep constants you will end up with general soln.

if you set them zero you will end up with particular soln.

ex $y'' + 2y' + y = \frac{e^{-t}}{t}$ by VOP

$$y_h = \lambda = -1^* \{e^{-t}, te^{-t}\} \quad y_h = c_1 e^{-t} + c_2 t e^{-t}$$

$$y_p = \mu_1 y_1 + \mu_2 y_2 \quad \text{then eqn of vop}$$

$$① e^{-t} \mu_1' + t e^{-t} \mu_2' = 0$$

$$② -e^{-t} \mu_1' + (e^{-t} - t e^{-t}) \mu_2' = \frac{e^{-t}}{t}$$

$$\begin{bmatrix} e^{-t} & t e^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{bmatrix} \begin{bmatrix} \mu_1' \\ \mu_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{-t}}{t} \end{bmatrix}$$

use
Cramer's rule

$$\mu_1' = \frac{\begin{vmatrix} 0 & t e^{-t} \\ e^{-t}/t & (1-t)e^{-t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & t e^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix}} = \frac{e^{-2t}}{e^{-2t}} = 1 \quad \mu_1' = 1 \quad \mu_1 = t + d_1$$

$$\mu_2' = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix}}{e^{-2t}} = \frac{e^{-2t}}{t e^{-2t}} = \frac{1}{t} \Rightarrow \mu_2' = \frac{1}{t} \Rightarrow \mu_2 = \ln|t|$$

$$y_p = \mu_1 y_1 + \mu_2 y_2$$

$$y_p = t e^{-t} + \ln|t| t e^{-t}$$

$$y_g = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + t e^{-t} + \ln|t| t e^{-t}$$

$$y_g = c_1 e^{-t} + c_3 t e^{-t} + t e^{-t} \ln|t|$$

ex $y'' + y = \tan t$ $\tan t \notin C^0$ reduction of order $\checkmark$ vop $\checkmark$ (constant coeff find y_h)

we'll solve by vop

$$y_h = \lambda^2 + 1 = 0 \quad \lambda = \pm i \quad y_h = c_1 \underbrace{\cos t}_{y_1} + c_2 \underbrace{\sin t}_{y_2}$$

$$y_p = u_1 \cos t + u_2 \sin t$$

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan t \end{bmatrix} \quad \text{solve for } u_1', u_2' \quad (\text{exercise})$$

(change of variable) $\cos = u \dots$

$$u_1 = \int \frac{\sin^2 t}{\cos t} dt = \underbrace{-\frac{1}{2} |\sec t + \tan t| - \sin t}_A + d_1''$$

$$u_2 = \int \sin t dt = \underbrace{-\cos t}_B + d_2''$$

$$y_p = A \cos t + B \sin t, \quad y_g = y_h + y_p$$

vop for higher order DE

$$(N) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1(t) y' + a_0(t) y = g(t)$$

$$(H) \quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1(t) y' + a_0(t) y = 0$$

$$y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \quad y_i: \text{lin indep soln (H)}$$

$$\text{assume } y_p = u_1(t) y_1 + u_2(t) y_2 + \dots + u_n(t) y_n$$

sub y_p in (N) and make some assumptions as we did in 2nd order case (force zero) to obtain corresp. eqs for vops

$$y_1 u_1' + y_2 u_2' + \dots + y_n u_n' = 0$$

$$y_1' u_1' + y_2' u_2' + \dots + y_n' u_n' = 0$$

$$\vdots$$

$$y_1^{(n-2)} u_1' + y_2^{(n-2)} u_2' + \dots + y_n^{(n-2)} u_n' = 0$$

$$y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \dots + y_n^{(n-1)} u_n' = g(t)$$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix}$$

Solve for u_i' , $i=1,2,\dots,n$ either by taking Ψ^{-1} or using Cramer's rule.

ex $y''' - y'' - y' + y = e^{2t}$ solve VOP

$y_h: \lambda^3 - \lambda^2 - \lambda + 1 = 0 \quad \lambda = 1^* \quad \lambda = -1$

$y_p: y_p = \mu_1 e^t + \mu_2 t e^t + \mu_3 e^{-t}$

$y_h = c_1 e^t + c_2 t e^t + c_3 e^{-t}$

VOP $\begin{bmatrix} e^t & t e^t & e^{-t} \\ e^t & (1+t)e^t & -e^{-t} \\ e^t & 2e^t + t e^t & e^{-t} \end{bmatrix} \begin{bmatrix} \mu_1' \\ \mu_2' \\ \mu_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}$

$\mu_1' = \frac{\begin{vmatrix} 0 & t e^t & e^{-t} \\ 0 & (1+t)e^t & -e^{-t} \\ e^{2t} & 2e^t + t e^t & e^{-t} \end{vmatrix}}{\det \varphi = 4e^t} \quad , \quad \mu_2' = \frac{|\varphi_2(t)|}{4e^t} \quad , \quad \mu_3' = \frac{|\varphi_3(t)|}{4e^t}$

$\mu_1' = \frac{1}{4} e^t (-2t-1) \Rightarrow \mu_1(t) = -\frac{1}{2} t e^t + \frac{1}{4} e^{-t} + d_1''^0$

$\mu_2' = \frac{1}{2} e^t \Rightarrow \mu_2(t) = \frac{1}{2} e^t + d_2''^0$

$\mu_3' = \frac{1}{4} e^{3t} \Rightarrow \mu_3(t) = \frac{1}{12} e^{3t} + d_3''^0$

$y_p(t) = e^t \left(-\frac{1}{2} t e^t + \frac{1}{4} e^{-t} \right) + \frac{t^2 e^t}{2} + \frac{1}{12} e^{2t} \quad , \quad y_g = y_h + y_p$

Abel's Identity for DE Thm

$y'' + a_1(t)y' + a_0(t)y = 0$

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \quad \text{satisfy}$

$w(y_1, y_2)(t) = \underbrace{w(y_1, y_2)(t_0)}_{\text{constant}} \underbrace{e^{-\int a_1(t) dt}}_{>0}$

remark: when DE is real-valued then

$w(y_1, y_2)$ is always either everywhere zero or always positive or always negative

at every $t \in I =$ given interval