

Ordinary Diff Fns

<https://stemjock.com/boyceode10e.htm>

Defn a diff equations is an equation which involves an unknown function (dependent var) e.g. y ,

and independent variable (s) e.g. x or t and the derivatives of unknown function wrt its independent var.

There are 2 types of DE $\begin{cases} \text{Ordinary ODE } y = y(x) \rightarrow \text{only one independent } y, y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2} \\ \text{partial PDE} \rightarrow \frac{\partial^2 y}{\partial x^2} \end{cases}$

ex $y' = 2x + 5$ $y = y(x)$ $y = ?$ ODE 1st order

$x^2 y'' - 2xy' + y = 0$ ODE 2nd order

$x^2 y''' + x^2 y'' + y' = 2x^2$ ODE 3rd order

$\frac{\partial^2 y}{\partial x_1} + 2 \frac{\partial^2 y}{\partial x_1 \partial x_2} \dots$ PDE

$u_t = u_{xx}$ $u = u(x, t)$ PDE (heat eqn)

$u_{xx} + u_{yy} = 0$ $u = u(x, y)$ PDE (Laplace Eqn)

Defn: the highest derivative of unknown function appearing in DE called the **order** of DE

ex $y'' + t^2(y')^4 = 2t$ 2nd order

Defn: the general form of n^{th} order ODE can be written as

$$F(t, \underset{\substack{\uparrow \\ \text{indep} \\ \text{var}}}{y}, \underset{\substack{\uparrow \\ \text{unknown}}}{y'}, y'', \dots, y^{(n)}) = 0 \quad y^{(n)} = \frac{d^n y}{dt^n}$$

remark: 1) moreover if $y^{(n)}$ can be left alone in $\textcircled{a}$ then

$y^{(n)} = f(t, y, y' \dots y^{(n-1)}) \rightarrow$ Standard form of n^{th} order ODE

2) the standard form of a DE

1st order $y' = f(t, y)$

2nd order $y'' = f(t, y, y')$

$$\text{ex } y' = \frac{y}{x^2} \rightarrow f(x, y)$$

$$\text{ex } y'' - 2xy' + 2x^2y + e^x = 0 \quad F(x, y, y', y'')$$

$$\text{or } y'' = 2xy' - 2x^2y - e^x \\ f(x, y, y')$$

Defn say we have n^{th} order DE

$$y^{(n)} = F(t, y, y' \dots y^{(n-1)}) \quad (1)$$

the conditions, $y(t_0) = y_0$, $y'(t_0) = y_1$, $y''(t_0) = y_2$, ..., $y^{(n-1)}(t_0) = y_{n-1} \rightarrow$ initial values
 $t \in [a, b]$ (2)
 t_0

initial conditions
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So the problem containing the (1) + (2) called IVP initial value problem

ex $y' = f(t, y)$ $y(t_0) = y_0$ IVP 1st order

$$y'' = f(t, y, y') \quad y(t_0) = y_0, \quad y'(t_0) = y_1 \quad \text{2nd order}$$

$$y' = y \quad y(0) = 1 \quad y(t) = ce^t \quad \text{general soln} \quad \text{put } t=0, y=1 \text{ find } c \rightarrow c=1$$

$$y(t) = e^t \rightarrow \text{unique soln}$$

Defn (3) the ODE $F(t, y, y' \dots y^{(n)}) = 0$ is called LINEAR DE if F is linear function of the variables $y, y', \dots, y^{(n)}$. That is there is no term n s.t. y multiplied with itself or its derivatives. Otherwise, its called NON-LINEAR ODE

ex $y' + 2y = 3x$ linear 1st ODE

$$y'y + 3x^2y = 0 \quad \text{non linear 1st ODE}$$

$$y'' + e^x y' + y^2 + 2x = 0 \quad \text{non linear 2nd ODE}$$

Solution of DE

Defn: consider $y^{(n)} = f(t, y, y' \dots y^{(n-1)}) \quad (1)$

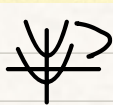
$$\text{IVP} \begin{cases} y(t_0) = y_0, \quad y'(t_0) = y_1, \dots \end{cases} \quad (2)$$

a) a function $y = \phi(t)$ that satisfies (1) of all pts t is an open interval I is called a "soln of DE (1)"

b) a function $y = \phi(t)$ which satisfies both (1) (2) is a soln of IVP

c) a soln containing constants which gives all possible solns for DE (1) is called "GENERAL SOLN" of DE (1)

ex $y' = 2x$ $y(x) = x^2 + C \rightarrow$ general soln of DE

 integral curves (solution curves)

however IVP $\begin{cases} y' = 2x \\ y(0) = 3 \end{cases}$ $y = x^2 + C$ $C = 3$ $y(x) = x^2 + 3$ unique soln for IVP

ex $y' = y \Rightarrow y = ce^x$  but $y(0) = 1$ so $C = 1$ so $y(x) = e^x$ unique IVP

ex $y'' + y = 0 \rightarrow y'' = -y$

$y = \phi_1(t) = \cos t$ is a soln

$y = \phi_2(t) = \sin t$ is a soln

$y = \phi_1(t) + \phi_2(t)$ is a soln also (general soln) of DE $= \phi_3(t) = y = C_1 \cos t + C_2 \sin t$

Remark: order of DE = # of constant in general soln

Defn: a soln y which can be expressed uniquely as a function of t only that is $y = y(t)$ is called EXPLICIT SOLN otherwise it's IMPLICIT SOLN. That is $f(t, y) = 0 \rightarrow$ implicit soln

ex $y(t) = e^{2t}$ is an explicit soln of $y' = 2y$

since $y'(t) = 2 \frac{e^{2t}}{1} \rightarrow y' = 2y$ interval of soln $I = (-\infty, \infty)$ e^{2t} defined $\forall t \in \mathbb{R}$

ex $y = x^2 - \frac{1}{x}$ is an explicit soln of $x^2 y' - 2y = 0$ $I = \mathbb{R} - \{0\}$

ex $x^2 + y^2 = 25$ $y = \pm \sqrt{25 - x^2}$ can't be written uniquely keep it closed form

$x^2 + y^2 = 25$ is implicit soln for DE $x + y y' = 0$

$y = \pm \sqrt{25 - x^2}$ but if $y = 0$ x has to be 5 but $x + y y' = 0$ not satisfy $I = (-5, 5)$

21 Feb 2024

review: $F(t, y, y' \dots y^{(n)}) = 0 \rightarrow$ general form of n -th order ODE

$y^{(n)} = f(t, y, y' \dots y^{(n-1)}) \rightarrow$ standard form of n -th order ODE

$y = \phi(t)$ called soln of DE

Solutions for 1st order ODE $y' = f(t, y)$

aim: classify DE into different types and develop methods for the soln of each type.

we have 2 ways for 1st order ODE

1) Derivative form $y' = f(t, y)$

2) Differential form $M(t, y) dt + N(t, y) dy = 0$

or if $f(t, y) = \frac{-M(t, y)}{N(t, y)} \rightarrow \frac{dy}{dt} = \frac{-M(t, y)}{N(t, y)}$

1) Separable DE

DE called separable iff f can be written as

$$f(t, y) = p(t) \cdot q(y)$$

$$y' = p(t) \cdot q(y) \rightarrow \frac{dy}{dt} = p(t) q(y) \rightarrow \frac{dy}{q(y)} = p(t) dt \xrightarrow{\text{integrate}} \int \frac{dy}{q(y)} = \int p(t) dt \quad [q(y) \neq 0]$$

remark if $q(y) = 0 \rightarrow y' = p(t) \cdot 0 = 0 \quad y' = 0 \rightarrow y = c \rightarrow \text{constant function}$

such a soln called equilibrium or rest or singular soln for DE

ex $y' = \frac{x^2}{1-y^2} \rightarrow p(x) = x^2, q(y) = 1-y^2$

$$\rightarrow \int (1-y^2) dy = \int x^2 dx$$

$$\boxed{y - \frac{y^3}{3} = \frac{x^3}{3} + c} \rightarrow \text{implicit form, there is "c" } \rightarrow \text{family of soln}$$

ex $y' + y^2 \sin x = 0$

$$\rightarrow y^2 \neq 0, y \neq 0$$

$$\frac{dy}{dx} = -y^2 \sin x \rightarrow \int \frac{dy}{y^2} = \int -\sin x dx \rightarrow \frac{-1}{y} = \cos x + c \rightarrow y = \frac{-1}{\cos x + c}$$

solve IVP

$$y' + y^2 \sin x = 0 \quad y = \frac{-1}{\cos x + c} \quad 0 = \frac{-1}{-1 + c} \quad \text{no soln for } c$$

y' nin tamamı 0 olursa

$y' = 0 \quad x' = 0$ soruya koy olursa al

try to put $y = 0 \rightarrow 0 = 0 \checkmark$ soln $\{0\}$, $y(\pi) = 0$ satisfies initial cond $y(x) = 0$ is a soln for IVP

ex $y' = \frac{2x}{1+2y}$ find the explicit soln

$$y(2) = 0$$

$$\int dy (1+2y) = \int dx \cdot 2x$$

$$y^2 + y = x^2 + C$$

$$0+0 = 2^2 + C \quad C = -4$$

$$y^2 + y = x^2 - 4$$

$$\left(y + \frac{1}{2}\right)^2 = x^2 - \frac{15}{4}$$

$$y + \frac{1}{2} = \pm \sqrt{x^2 - \frac{15}{4}}$$

which one? Initial cond
(+) one $\rightarrow y(2) = 0$

explicit soln of IVP $y = \frac{-1}{2} + \sqrt{x^2 - \frac{15}{4}}$

2) (Algebraically) Homogenous DE

Defn: $h(x,y)$ called "homogenous of degree n " if

$$\frac{x^2}{1+y^3} \rightarrow \text{not homogenous} \quad \frac{\lambda^2 x^2}{1+\lambda^3 y^3}$$

$$h(\lambda x, \lambda y) = \lambda^n h(x,y) \quad \forall \lambda > 0$$

$\forall x,y$ where $h(x,y)$ is defined.

ex $h(x,y) = \frac{x^2}{x^2+y^2}$

$$h(\lambda x, \lambda y) = \frac{(\lambda x)^2}{(\lambda x)^2 + (\lambda y)^2} = \frac{\lambda^2 x^2}{\lambda^2 (x^2 + y^2)} \rightarrow \frac{x^2}{x^2 + y^2} = h(x,y)$$

Defn: $y' = f(x,y)$ is called algebraically homogenous if f is a homogenous function of degree 0

that is $f(\lambda x, \lambda y) = f(x,y)$

remark

1) let $f(x,y) = -\frac{M(x,y)}{N(x,y)}$

$$\rightarrow y' = f(x,y) \rightarrow M(x,y) dx + N(x,y) dy = 0 \quad (*)$$

if M and N are homg funct of same degree then eqn $(*)$ is called homogenous.

2) $y' = f(x,y)$ is algebr hom iff $f(x,y)$ can be expressed as $\frac{x}{y}$ or $\frac{y}{x} \rightarrow f(x,y) = h\left(\frac{y}{x}\right)$ or $f\left(\frac{x}{y}\right)$

to solve homogenous DE use substitution $V = \frac{dy}{dx}$ or $v = \frac{y}{x}$

use $v = \frac{y}{x} \rightarrow y = vx$ diff wrt x

$$y' = v + xv' \quad (v = v(x))$$

write $y' = f(x, y)$ in terms of v

$$f(x, y) = h\left(\frac{y}{x}\right) \rightarrow v + xv' = h(v) \rightarrow v' = \frac{1}{x}(h(v) - v)$$

separable

$$\rightarrow \int \frac{dv}{h(v) - v} = \int \frac{1}{x} dx$$

$$\text{ex } \underbrace{(y^2 + xy + x^2)}_M dx - \underbrace{x^2 dy}_N = 0$$

$y^2 \rightarrow x^2$ $xy \rightarrow x^2$ $x^2 \rightarrow x^2$ M and N have same degree $\rightarrow$ DE algeb. hom.

$$\frac{dy}{dx} = \frac{y^2 + xy + x^2}{x^2} \rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 1 = h\left(\frac{y}{x}\right)$$

$$v = \frac{y}{x} \quad y = vx \quad y' = v + xv'$$

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1$$

$$v + xv' = v^2 + v + 1 \quad xv' = v^2 + 1 \rightarrow v' = \frac{1}{x}(v^2 + 1) \quad \text{separable}$$

$$\frac{dv}{dx} = \frac{1}{x}(v^2 + 1) \rightarrow \int \frac{dv}{v^2 + 1} = \int dx \frac{1}{x} \rightarrow \arctan v = \ln|x| + c$$
$$\arctan\left(\frac{y}{x}\right) = \ln|x| + c$$

$$\text{ex } y' = \frac{y^2 + x^2}{yx} \xrightarrow{f(x,y)} y(1) = 2$$

not separable $f(x, y) = \frac{y}{x} + \frac{x}{y} \rightarrow \frac{y}{x} + \frac{1}{\frac{y}{x}} \rightarrow h\left(\frac{y}{x}\right) \rightarrow$ algeb. hom.

$$\frac{y}{x} = v \quad y = xv \quad y' = v + xv'$$

$$\text{DE: } v + xv' = v + \frac{1}{v} \rightarrow v' = \frac{1}{x} \cdot \frac{1}{v} \quad v' \cdot v = \frac{1}{x} dx \rightarrow v dv = \frac{1}{x} dx$$
$$\frac{v^2}{2} = \ln|x| + c$$

$$\frac{y^2}{2x^2} = \ln|x| + c \quad \text{find } c \text{ with } y(1) = 2$$

$$\frac{4}{2} = 0 + c \quad c = 2 \quad \frac{y^2}{2x^2} = \ln|x| + 2 \quad \text{soln } \checkmark \quad \text{if its explicit } \rightarrow y = \pm \sqrt{4x^2 + 2x^2 \ln|x|}$$

exercise $y' = \frac{x+3y}{x-y}$, $y(1) = -1$ Solve IVP (hint divide by x)

ex $y' + \frac{1}{x}y = \ln(x)$ [linear]

$y' = \ln x - \frac{1}{x}y$ not sepr. not homog.

how to say something about soln of DE without solving DE

- 1) look at direction graph
- 2) existence and uniqueness thm.

1) Solution curves and Direction Field $y' = f(x, y)$

analytically: find $y = \Phi(x)$ satisfy DE

geometrically: direction fields, solution (integral) curves

ex $y' = -y$ $\frac{dy}{y} = -dx \rightarrow \ln|y| = -x + c_1$ $y = e^{-x+c_1} \rightarrow e^{-x} \cdot C$

- 1) pick a slope $C \in \mathbb{R}$ constant
- 2) plot $f(x, y) = C$ curve since $y' = \text{slope} = C$
- 3) Draw small line segments with slope C on curve $f(x, y) = C$

$y' = -y \rightarrow f(x, y) = -y = C$

IC $y(0) = 1 \rightarrow y(x) \equiv 0$ (equilibrium soln)

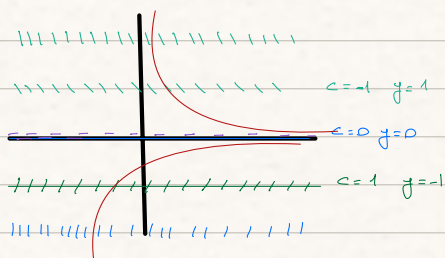
$C=0$ $y=0$

$C=1$ $y=-1$

$C=2$ $y=-2$

$C=-1$ $y=1$

$C=-2$ $y=2$

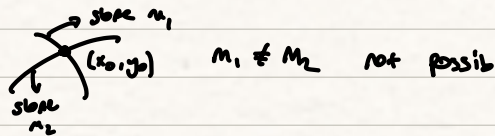


Defn $y' = 0 \Rightarrow y = L$, L : constant is called equilibrium (rest) soln

an equilibrium soln is called asymptotically stable as $x \rightarrow \infty$ if all other solns are approaching to the equilibrium soln. Otherwise its called asymp. unstable soln

ex ex 1 $y(x)=0 \Rightarrow$ asym stable

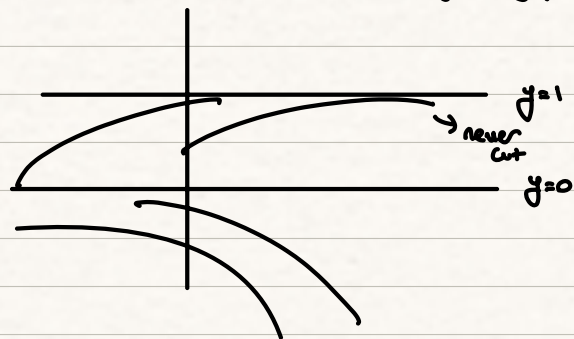
① two soln curves can't cross at an angle (that is $f(x,y)$ can not have 2 diff values at a point)



② two soln curves can not be tangent to each other (due to uniqueness)

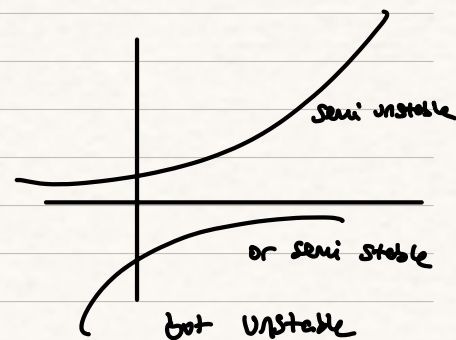
ex $y' = \underbrace{y(1-y)}_{f(x,y)}$

$f(x,y) = y(1-y) = 0$ $y' = \begin{matrix} 0 & 1 \\ - & + \\ y & \downarrow \end{matrix}$ $y(x)=0$ $y(x)=1$ are equilibrium soln.



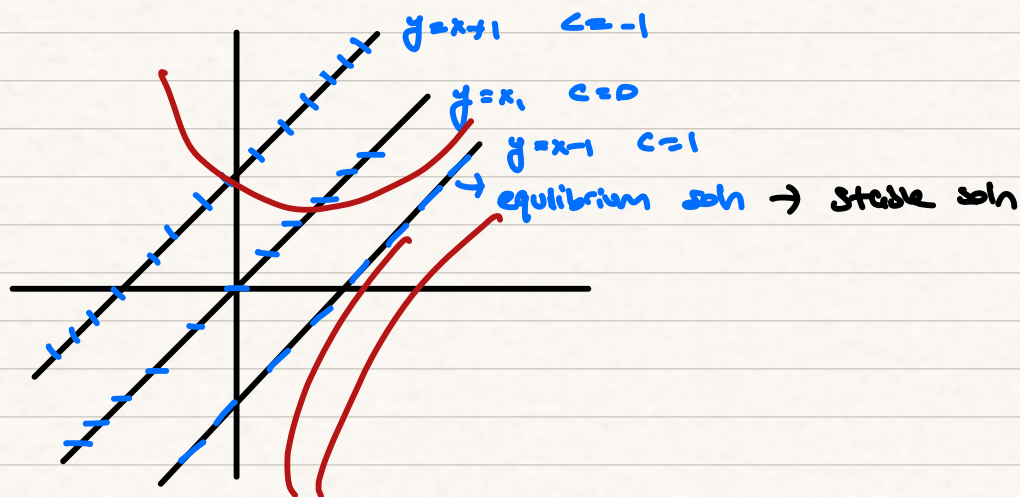
$x \rightarrow \infty$ all soln around $y=1$ approach the equilibrium soln $y=1$ so its asym stable

$y(x)=0$ is asym unstable



Defn: if $f(x,y) = f(y)$ then the DE $y' = f(y)$ is called autonomous DE

ex $y' = \underbrace{x-y}_{f(x,y)} \quad f(x,y) = x-y = c \rightarrow y = x-c$



Existence and Uniqueness Thm

consider the IVP $y' = f(x, y)$
 $y(x_0) = y_0$

a) existence if $f(x, y)$ is cont on a rectangle R containing (x_0, y_0)

$R = \{(x, y) \mid |x - x_0| < \alpha, |y - y_0| < \beta\}$ then IVP has at least 1 soln on small interval $|x - x_0| < h$ ^{$h > 0$ small}

b) uniqueness if $\frac{\partial f}{\partial y}$ is also cont on R then the IVP has a **unique soln** pass through (x_0, y_0)

ex $y' = y, y(0) = 1$

$f(x, y) = y$ is cont $\forall (x, y) \in \mathbb{R}^2$

$$R = (-\infty, \infty) \times (-\infty, \infty) = \mathbb{R}^2$$

$x \qquad y$

so IVP has at least one soln in $\mathbb{R}^2 \ni (0, 1)$

$\frac{\partial f}{\partial y} = 1$ is cont. $\forall (x, y) \in \mathbb{R}^2$ so IVP has a unique soln in a rectangle R containing $(0, 1)$

ex $y' = xy^{1/2}, y \geq 0$ a) $y(0) = 0$ discuss E and U
b) $y(0) = 1$ IVP

a) $y' = xy^{1/2}$, $y(0) = 0$ $f(x, y)$ cont. $\forall x \in \mathbb{R}$ and $\forall y \geq 0$
 $\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$ is cont $\forall x \in \mathbb{R}$ and $\forall y > 0$

existence, since $f(x, y)$ cont $\forall (x, y)$ st $y \geq 0$, its possible to draw rectangle containing initial point $(0, 0)$ so that IVP has at least 1 soln

uniqueness, there is no rectangle containing initial pt. $(0, 0)$ E, U fails

say nothing about number of solns, that is soln may be unique or not. Check by solving

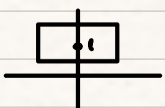
$y' = xy^{1/2}$ separate

$$\int \frac{dy}{y^{1/2}} = \int x dx \quad \text{if } y \neq 0, \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + c, \quad y = \left(\frac{x^2}{4} + \frac{1}{2}c\right)^2 \quad \text{find } c \text{ use IVP } y(0) = 0$$

$$0 = \left(0 + \frac{1}{2}c\right)^2 \quad c = 0$$

soln $\Rightarrow y = \frac{x^4}{16}$ is the soln, possibly $(0, 0)$, consider $y(x) = 0$ is also a soln?

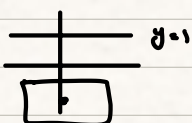
$$y' = xy^{1/2}, y(x) = 0, \quad 0 = x \cdot 0^{1/2} \checkmark \quad y(0) = 0 \checkmark$$

b) $y' = x y''$ $y(0) = 1$  its possible to draw rectangle R containing $(0,1)$

s.t both f and $\frac{\partial f}{\partial y}$ cont on R , largest possible rectangle $R = (-\infty, \infty) \times (0, \infty)$

ex what is the max rectangular region to apply the E and U Thm to $y' = \frac{3x^2 + 4x + 2}{2(y-1)}$, a) $y(0) = -1$ b) $y(0) = 1$

existence $f(x,y)$ cont $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $y \neq 1$

a) $y(0) = -1$ 

Since R contains $(0,-1)$ exists so by E and U exist

$\frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y-1)^2}$ is cont $\forall x \in \mathbb{R}$ $\forall y \neq 1$ so soln is unique as well with the choice of max rectangle

$$R = (-\infty, \infty) \times (-\infty, 1)$$

b) E and U fails. solve IVP check it.

$$\int 2(y-1) dy = \int (3x^2 + 4x + 2) dx$$

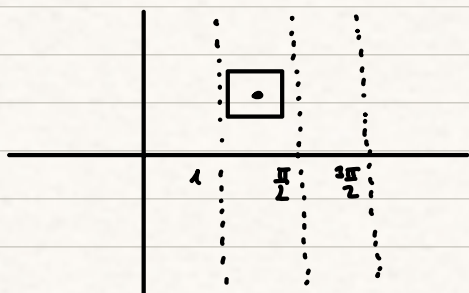
$$(y-1)^2 = x^3 + 2x^2 + 2x + C$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C} \quad \text{put } y=1 \quad x=0 \quad \text{find } C, C=0$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x} \quad \text{both of them are true IVP we have 2 solns}$$

ex $y' = \frac{1}{2t+1} \tan(t) \sqrt{y}$, find the possible largest rectangle region contains $(\pi/3+1)$ where E and U is applicable

$f(t,y)$ cont. when $t > 0$, $t \neq 1$, $\forall t \cos(t) \neq 0$, $t \neq (2k+1)\frac{\pi}{2}$ and $y > 0$



$$\frac{\partial f}{\partial y} = \frac{1}{4t+1} \tan(t) \frac{1}{2\sqrt{y}} \quad \text{cont } t > 0 \quad t \neq 1 \quad \forall t \neq (2k+1)\frac{\pi}{2} \quad k=0, \pm 1. \quad \text{and } y > 0$$

$$R = (-1, \frac{\pi}{2}) \times (0, \infty)$$

Ex 1) $y'' + 2xy' + e^x y = \frac{e^x \ln x}{g(x)}$

non-hom. 2nd order linear DE

2) $(y' y'') + e^x y' = x^2$

non linear

3) $(\arctan y \cdot y''') + (2x^2 + 1) + \ln x y = \sin 3x$

non linear

1st order Linear DEs

Any eqn. of the form

(1) $y' + p(x)y = q(x)$

is called linear 1st order DE.

when $p(x)$ and $q(x)$ are continuous.

For homogeneous case when $q(x) = 0$:

Then DE (1) becomes

$y' + p(x)y = 0$ which is separable

$\int \frac{dy}{y} = \int -p(x) dx$

$\ln|y| = -\int p(x) dx \Rightarrow y = e^{-\int p(x) dx}$
Solu. corresponding to hom. DE

Linear DEs

Defn: 1) The DE $F(x, y, y', \dots, y^{(n)}) = 0$

is linear if F is linear in y and its derivatives. That is, F does NOT contain the multiplication of y and its derivatives with themselves. Then, the standard form of n th order linear DE is:

(*) $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$

if $a_n(x) \neq 0$.

2) The DE (*) is homogeneous if $g(x) = 0$. Otherwise if $g(x) \neq 0$ then DE (*) is called non-homogeneous.

For non homogeneous case when $q(x) \neq 0$

method of integrating factor

find a function $\mu(x)$ (integrating factor) s.t. when we multiply DE (1) with $\mu(x)$ then the LHS of eqn will be derivative of $\mu(x)y(x)$

$y' + p(x)y = q(x)$ multiplying with μ

$\mu y' + \mu p y = \mu q$

$\int (\mu y)' = \int \mu q dx$

$\mu y = \int \mu q dx + C$

$y(x) = \frac{1}{\mu} \int \mu q dx + \frac{C}{\mu} \Rightarrow$ general soln for 1st order

particular soln y_p

soln corresponding homogeneous part y_c

Question how to find $\mu(x)$?

use equation $(\mu y)' = \mu y' + \mu p y$

$$\mu' y + \cancel{\mu y'} = \cancel{\mu y'} + \mu p y$$

$$\mu' y = \mu p y \rightarrow \mu' = \mu p \rightarrow \int \frac{d\mu}{\mu} = \int p(x) dx$$

so $\ln|\mu| = \int p(x) dx + c \rightarrow |\mu| = \left(e^c\right) \cdot e^{\int p(x) dx}$

$$\mu = c \cdot e^{\int p(x) dx}$$

(we need to find min 1 μ so WLOG pick $\mu = e^{\int p(x) dx}$)

ex $y' - \frac{1}{x} y = \frac{\ln x}{x}$ linear 1st order

$$\mu = e^{\int p(x) dx} \rightarrow e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} \quad (\text{don't find } c \text{ when we do integrating factor})$$
$$= -\frac{1}{|x|} = \frac{1}{x} \quad x > 0$$

multiply DE by $\mu(x) = \frac{1}{x}$

$$\frac{1}{x} \cdot y' - \frac{1}{x^2} y = \frac{1}{x} \ln x$$

$$\int \left(\frac{1}{x} y\right)' = \int \frac{1}{x} \ln x$$

general soln $\rightarrow$ particular soln $\rightarrow$ homogeneous soln

$$y = \frac{x(\ln x)^2}{2} + c x$$

$$\mu = \frac{\ln x}{x} \rightarrow \int u dv \rightarrow \frac{u^2}{2} + c$$

$$\Rightarrow y_g = y_p + y_c \quad \checkmark$$

Existence and Uniqueness for 1st Order

$$y' = f(x, y)$$

$$y' = -p(x)y + q$$

if the functions $p(x)$ and $q(x)$ are cont. on interval I containing x_0 then there is a unique soln for the IVP

$$\left. \begin{array}{l} y' + p(x)y = q(x) \\ y(x_0) = y_0 \end{array} \right\} \text{ IVP in interval } I$$

soln interval

ex determine (no solving) the largest interval where IVP has a unique soln.

$$1) x \frac{dy}{dx} + \frac{2x+1}{x+1} y = x-1$$

$$\frac{dy}{dx} + \frac{2x+1}{x(x+1)} y = \frac{x-1}{x}$$

p is cont when $x \neq 0$ $x \neq -1$
 q is cont when $x \neq 0$

by E and U then the DE has unique solns on

intervals not containing $0, -1$.



$$I = (-\infty, -1)$$

$$II = (-1, 0)$$

$$III = (0, \infty)$$

a) let IC $y(1) = -1$ $\rightarrow$ III region, the largest is $(0, \infty)$

b) let IC $y(\frac{-1}{2}) = 2$ II $(-1, 0)$

let IC $y(-2) = 3$ I $(-\infty, -1)$

Find soln $\frac{dy}{dx} + \frac{2x+1}{x(x+1)} y = \frac{x-1}{x}$

$M = e^{\int p(x) dx} = e^{\int \frac{2x+1}{x(x+1)} dx} = e^{\ln(x^2+x)} = x^2+x$

Mul. DE with $M = x^2+x$

$$(x^2+x)y' + (2x+1)y = \frac{x-1}{x}$$

$$\left[(x^2+x)y \right]' = (x-1)(x+1) dx$$

$$(x^2+x)y = \frac{x^3}{3} - x + C$$

$$y(x) = \frac{1}{3} \frac{x^3}{(x^2+x)} - \frac{x}{x^2+x} + \frac{C}{x^2+x}$$

Dom $y = \mathbb{R} - \{0, -1\}$

a) $y(1) = -1$ $-1 - \frac{4}{6} = \frac{C}{2}$

$$\Rightarrow -1 = \frac{1}{6} + \frac{1}{2} + \frac{C}{2}$$

$$C = -\frac{6}{3}$$

$\Rightarrow$ Unique soln.

$$y(x) = \frac{1}{3} \frac{x^2}{x+1} - \frac{x}{x^2+x} - \frac{4}{3} \frac{1}{x^2+x}$$

Exm Determine (without solving) the largest interval where IVP has a unique soln.

1.) $x \frac{dy}{dx} + \frac{2x+1}{x+1} y = x-1$

Exercise find the largest interval where IVP has unique soln.

$$\cos x + y' = \sin x y - \sqrt{x-1}, \quad y(3/2) = 0$$

Remark

1) the soln of $y' + p(x)y = q(x)$ is $y(x) = \underbrace{\frac{1}{\mu}}_{\text{IF}} \underbrace{\int \mu q dx}_{\text{IP}} + \underbrace{\frac{C}{\mu}}_{y_c = y_h}$

Substitution

$$y' = \sqrt{x+y} - 1 \quad \text{not sep. not linear}$$

the problem is $\sqrt{x+y}$

$$\text{say } v = \sqrt{x+y} \Rightarrow v' = \frac{x+y}{2\sqrt{x+y}} \rightarrow 2vv' - 1 = y'$$

substitute in DE in terms of v

$$y' = \sqrt{x+y} - 1$$

$$2vv' - 1 = v - 1 \rightarrow v' = \frac{1}{2} \rightarrow dv = \frac{1}{2} dx$$

ex bernoulli eqn non linear

$$v = \frac{1}{2}x + c \rightarrow \sqrt{x+y} = \frac{1}{2}x + c, \quad y = -x + \left(\frac{1}{2}x + c\right)^2$$

$$(*) \quad y' + p(x)y = q(x)y^r \quad r \in \mathbb{R}$$

i) $r=0$, linear

ii) $r=1$, sepr.

iii) $r \neq 0,1 \Rightarrow$ non linear use substitution $v = y^{-r+1}$

multiply DE with y^{-r}

$$y^{-r} \cdot y' + p(x) y^{-r+1} = q(x) \quad (\text{**})$$

$$\text{use } v = y^{-r+1}$$

$$v' = (-r+1) y^{-r} \quad \text{subs in } (*) \rightarrow \frac{v'}{-r+1} + p(x)v = q(x)$$

$$v' + (1-r)p(x)v = (1-r)q(x) \quad \text{"linear"} \quad \mu = e^{\int (1-r)p(x) dx} \quad \text{we know how to solve}$$

ex $y' - 2xy = y^3 e^{-2x^2}$ Bernoulli $r=3$

use $v = y^{-r+1} = y^{-2}$

multiply with $y^r \rightarrow y^{-3}$

$$y^3 y' - 2xy^2 = e^{-2x^2} \quad v = y^{-2} \rightarrow v' = -2y^{-3} y'$$

$$\frac{v'}{-2} - 2xv = e^{-2x^2} \rightarrow v' + 4xv = -2e^{-2x^2} \quad \text{linear}$$

$$\mu = e^{\int 4x dx} = e^{2x^2}$$

$$\underbrace{e^{2x^2} v' + 4x e^{2x^2} v}_{\int (e^{2x^2} v)'} = \int -2 dx \Rightarrow e^{2x^2} v = -2x + C$$

$$v = -2x e^{-2x^2} + C e^{-2x^2}$$

$$\boxed{y^{-2} = -2x e^{-2x^2} + C e^{-2x^2}} \quad \text{implicit form}$$

ex Riccati eqn non linear

$$y' = p(x)y + q(x)y^2 + r(x), \quad r(x) \neq 0$$

this eqn can't be solved unless a partic soln y_p is known

let y_p is given, then use a new variable $z = y - y_p$

$$z' + y_p' = p(x)(z + y_p) + q(x)(z^2 + y_p^2 + 2zy_p) + r(x) \quad z' = y' - y_p' \text{ and by DE}$$

$$z' = [p(x) + 2y_p(x)q(x)]z + q(x)z^2$$

$$\underbrace{z' + y_p'}_{\checkmark} = \underbrace{p(x)y_p + q(x)y_p^2 + r(x)}_{\text{cancel}} + 2p(x)z + q(x)z^2 + 2zy_p$$

cancel

$$z' - h(x)z = q(x)z^2 \rightarrow \text{Bernoulli with } r=-2 \quad \text{use } v = z^{-2+1} = z^{-1}$$

Riccati's Eqn

$$y' = p(x)y + q(x)y^2 + r(x), \quad r(x) \neq 0$$

Using substitution

(1) $z = y - y_p$, y_p is a solution to DE.
results in

$$z' = [p(x) + 2y_p(x)q(x)]z + q(x)z^2$$

which is Bernoulli eqn. with $r=2$

Use subs. (2) $v = z^{1-r} = \frac{1}{z} \Rightarrow z = \frac{1}{v}$

(1) and (2) $\Rightarrow \frac{1}{v} = y - y_p \Rightarrow$
(3) $y = \frac{1}{v} + y_p$

If we use (3) we can obtain soln. in one step

Exm $y' = x^2 - y^2 + 1$, $y_p(x) = x$ is a soln. for DE.

Subs. $y = y_p + \frac{1}{v} = x + \frac{1}{v}$

$$y' = 1 - \frac{v'}{v^2}$$

DE: $y' = x^2 - y^2 + 1$

$$1 - \frac{v'}{v^2} = x^2 - (x + \frac{1}{v})^2 + 1$$

$$v' = 2xv + 1 \quad \text{"linear"}$$

$$\mu(x) = e^{\int -2x dx} = e^{-x^2}$$

$$e^{-x^2} v' - 2xe^{-x^2} v = e^{-x^2}$$

$$\int [e^{-x^2} v]' = \int e^{-x^2} dx$$

$$e^{-x^2} v = \int e^{-x^2} dx + C$$

$$v = e^{x^2} \int e^{-x^2} dx + C e^{x^2}$$

$$\frac{1}{y-x}$$

Exact DEs and integrating factors

Defn: The DE

$$M(x,y)dx + N(x,y)dy = 0$$

is called exact DE for continuous M and N

if there is a continuously differentiable

function $F(x,y)$, which is called potential function, st.

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

Then the general soln. of the DE is given by

$$F(x,y) = C, \quad C: \text{constant.}$$

Thm Consider DE $M(x,y)dx + N(x,y)dy = 0$

where M, N and $\frac{\partial M}{\partial y} = M_y$, $\frac{\partial N}{\partial x} = N_x$

are continuous in a rectangular region

$R = (a,b) \times (c,d)$ then the DE is exact in R if

$$M_y = N_x \quad \text{"exactness condition"}$$

that is $\exists$ a potential function $F(x,y)$ satisfy

$$F_x = M \quad F_y = N$$

iff M and N satisfy exactness condition

ex solve $(ye^x + 2e^x + y^2) dx + (e^x + 2xy) dy = 0$, $y(0) = 6$

M N

$$\frac{\partial M}{\partial y} = M_y = e^x + 2y = e^x + 2y = N_x = \frac{\partial N}{\partial x}$$

M, N, M_y, N_x cont everywhere $\mathbb{R}^2 \Rightarrow$ DE exact, $\exists$ a potential function $F(x,y)$ st

$$\textcircled{1} \frac{\partial F}{\partial x} = M = ye^x + 2e^x + y^2$$

$$\textcircled{2} \frac{\partial F}{\partial y} = N = e^x + 2xy$$

start by $\textcircled{1}$ integrate $\textcircled{1}$ wrt x

$$F(x,y) = ye^x + 2e^x + y^2x + h(y) \rightarrow \text{func } y \text{ constant wrt } x$$

now diff F wrt y and equate the result eqn $\textcircled{2}$

$$\frac{\partial F}{\partial y} = \underline{e^x} + \underline{2yx} + h'(y) \xrightarrow{\frac{dh}{dy}} = \underline{e^x} + \underline{2xy} \quad \Rightarrow h'(y) = 0 \rightarrow h(y) = C \text{ (constant)}$$

$$\text{so } F(x,y) = ye^x + 2e^x + y^2x + C_1 \quad (\text{NOT a SOLN})$$

$$\text{so } F(x,y) = C \rightarrow ye^x + 2e^x + y^2x = d \quad (\text{SOLN})$$

$$\text{now find } d \text{ with } y(0) = 6 \quad 6 + 2 = d \quad d = 8$$

$$\boxed{ye^x + 2e^x + y^2x = 8} \quad \text{SOLN EXPLICIT}$$

ex $(2x + 3y^2) dx + (6xy + 3y^2 + 4) dy = 0$

M N

$M_y = 6y = N_x$ all cont in $\mathbb{R}^2$

① $F_x = M = 2x + 3y^2$

② $F_y = N = 6xy + 3y^2 + 4$

integrate ② wrt y

$F(x, y) = 3xy^2 + y^3 + 4y + g(x) \rightarrow !!$

Diff F wrt x and = it ①

$\frac{\partial F}{\partial x} = 3y^2 + g'(x) = 2x + 3y^2 \rightarrow g'(x) = 2x$

$g(x) = x^2 + d$

$F(x, y) = 3xy^2 + y^3 + 4y + x^2 + d$ NOT A SOLN

$3xy^2 + y^3 + 4y + x^2 = C$ SOLN

exM $(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0$ not exact

M N

$M_y = 3 + 8xy \neq 2 + 6xy = N_x$ DE NOT EXACT?

let's multiply DE by $\mu(x, y) = x^2y$ then

$(3x^2y^2 + 4x^3y^3) dx + (2x^3y + 3x^4y^2) dy = 0$

R Q

$R_y = 6x^2y + 12x^3y^2 = 6x^2y + 12x^3y^2 = Q_x$ thus DE is now exact

find such a function $\mu(x, y)$ making DE exact after multiplication is integrating factor

How do we find $\mu(x, y)$?

Integrating Factors

Defn Consider the DE $M(x,y)dx + N(x,y)dy = 0$

• function $\mu(x,y) \neq 0$ (integrating factor) of DE if $\neq$ not exact but

$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ ~~is~~ is exact.

to find $\mu(x,y)$ we know ~~is~~ is exact then $(\mu M)_y = (\mu N)_x$ "exactness cond"

~~is~~ $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ PDE we DO NOT HAVE TO solve

however we have the soln for some special cases of $\mu(x,y)$

case 1 μ is only x $\mu = \mu(x)$

then $\mu_x = \mu' = \frac{d\mu}{dx}$, $\mu_y = 0$ subs in ~~is~~ $\rightarrow \mu M_y = \mu' N + \mu N_x$ (separable)

$$\mu' N = \mu (M_y - N_x) \rightarrow \text{not zero}$$

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N} \quad \underline{\text{is exact}}$$

only x has to be only x

, if not our assumption $\mu(x)$ is wrong!
try other ways

also if both sides are "x" terms only
we can find $\mu(x)$ by integrating ~~is~~

$$\ln |\mu| = \int \frac{M_y - N_x}{N} dx \quad (\text{woc})$$

$$|\mu| = e^{\int \frac{M_y - N_x}{N} dx} + C \quad (\text{pick (+) one})$$

for simplicity choose

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

when $\mu = \mu(x)$

case 2 μ is only y $\mu = \mu(y)$

then $\mu_x = 0$, $\mu_y = \frac{d\mu}{dy} = \mu'$

~~is~~ $\rightarrow \mu' M + \mu M_y = 0 \cdot N + \mu N_x$

$$\mu' M = \mu (N_x - M_y) \rightarrow \frac{\mu'}{\mu} = \frac{N_x - M_y}{M}$$

only y has to be only y

$$\int \frac{\mu'}{\mu} dy = \int \frac{N_x - M_y}{M} dy$$

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

only y contain

Case 3 if μ is x and y $\mu(x, y)$ then it must be given in this level course

ex $\underbrace{2xy}_{M} dx - \underbrace{(2x^2 + y)}_N dy = 0$ not separable check exact

$M_y = 2x \neq N_x = -4x$ not exact DE find μ

assume μ is x only

check $\frac{M_y - N_x}{N} = \frac{2x + 4x}{-2x^2 - y}$ not x only Stop we are wrong

assume μ is y only $\mu(y)$

$\frac{N_x - M_y}{M} \rightarrow \frac{-4x - 2x}{2xy} \rightarrow \frac{-6x}{2xy} = \frac{-3}{y}$ nice only y we are correct $\therefore$

$\mu = e^{\int \frac{-3}{y} dy} = e^{-3 \ln y} \rightarrow e^{\ln y^{-3}} \rightarrow \frac{1}{y^3} = \mu(y)$

we found $\mu(y)$

$\mu 2xy dx - \mu (2x^2 + y) dy = 0$ is exact with $\mu = \frac{1}{y^3}$

$\frac{\partial F}{\partial x} = \mu 2xy = \frac{2xy}{y^3} = \frac{2x}{y^2}$ integr wrt x $F(x, y) = \frac{x^2}{y^2} + h(y) = -\frac{2x^2}{y^3} + h'(y)$ ^{by 2} $= \frac{-2x^2}{y^3} - \frac{1}{y^2}$
diff wrt y

$\frac{\partial F}{\partial y} = -\mu(2x^2 + y) = -\frac{2x^2 + y}{y^2}$

$h'(y) = \frac{-1}{y^2}$, $h(y) = \frac{1}{y} + C$

then soln to DE (both multiplied with μ or original)

$F(x, y) = C$, $\frac{x^2}{y^2} + \frac{1}{y} + C_1 = C$ $\frac{x^2}{y^2} + \frac{1}{y} = C$ soln

ex $(x^2 e^y - 1) dx + (y^3 e^y + x) dy$

Exact DE and Integrating factors
 $Mdx + Ndy = 0$

$M_y = N_x$ and M, N, M_y, N_x
 are all continuous.

$\mu Mdx + \mu Ndy = 0$ Find μ st.

$$(\mu M)_y = (\mu N)_x$$

Exm. Show that DE is not exact.
 but becomes exact when multiplied
 with the integrating factor. Use this
 to find soln

$$a) \underbrace{(2yxe^x + 2x^2ye^x)}_M dx + \underbrace{(6y^3x - 2x^2e^x)}_N dy = 0$$

$$\mu = \frac{1}{xy^2}$$

$$M_y \stackrel{?}{=} N_x \quad M_y = 2xe^x + 2x^2e^x \quad N_x = 6y^3 - 4xe^x - 2x^2e^x \quad \Rightarrow \text{DE is not exact.}$$

$$\mu M = \frac{2}{y}e^x + \frac{2x}{y}e^x, \quad (\mu M)_y = -\frac{2}{y^2}e^x - \frac{2x}{y^2}e^x$$

$$\mu N = 6y - \frac{2xe^x}{y^2} \quad (\mu N)_x = -\frac{2}{y^2}e^x - \frac{2x}{y^2}e^x \quad \Rightarrow \text{DE becomes exact.}$$

Then $\exists F$ st.

$$① F_x = \mu M = \frac{2}{y}e^x + \frac{2x}{y}e^x$$

$$② F_y = \mu N = 6y - \frac{2xe^x}{y^2}$$

Int. ② wrt y .

$$F(x, y) = 3y^2 + \frac{2xe^x}{y} + h(x) \quad \text{diff. wrt } x$$

$$F_x = \frac{2e^x}{y} + \frac{2xe^x}{y} + h'(x) \stackrel{①}{=} \frac{2e^x}{y} + \frac{2xe^x}{y} \Rightarrow h'(x) = 0 \Rightarrow h(x) = C_1$$

Then soln. to DE is

$$F(x, y) = d \quad d: \text{constant}$$

$$3y^2 + \frac{2xe^x}{y} + C_1 = d$$

$$\boxed{3y^2 + \frac{2xe^x}{y} = C}, \quad C = d - C_1$$

b) Exercise

$$\left(\frac{\sin y}{y} - 2e^{-x} \sin x \right) + \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) y' = 0$$

$$\mu(x, y) = ye^x$$

Remark 1st order linear DE

$$y' + p(x)y = q(x) \quad \text{rewrite}$$

$$\frac{1}{N} dy + \underbrace{\frac{(p(x)y - q(x))}{M}}_M dx = 0$$

$$M_y = p(x) \neq 0 = N_x \quad \text{not exact.}$$

Find μ st μ times DE becomes exact.

$$\mu M = \mu(p(x)y - q(x)) \quad (\mu M)_y = (\mu N)_x$$

$$\mu N = \mu$$

Assume μ is a function of x -only.

$$\text{Check: } \frac{M_y - N_x}{N} = \frac{p(x) - 0}{1} = p(x) \quad \text{a function of } x \text{-only}$$

So our assumption $\mu = \mu(x)$ is correct.

$$\text{Then, } \mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int p(x) dx}$$

Observe that this is the same integrating factor as we obtain before.

$$\text{Ex} \quad \underbrace{(x^2 e^y - 1)}_M dx + \underbrace{(y^3 e^y + x)}_N dy = 0$$

$$M_y = x^2 e^y \neq 1 = N_x \quad \text{DE is not exact}$$

Assumption 1 $\mu = \mu(x)$

$$\text{Check } \frac{M_y - N_x}{N} = \frac{x^2 e^y - 1}{y^3 e^y + x} \quad \text{not a funct. of } x \quad \text{only } y \downarrow$$

Assumption 2 $\mu = \mu(y)$

$$\text{Check } \frac{N_x - M_y}{M} = \frac{1 - x^2 e^y}{x^2 e^y - 1} = -1 \quad \text{constant} \\ \text{can be thought as a function } y.$$

Then assumption 2 is working. That is

$$\mu(y) = e^{\int -1 dy} = e^{-y}$$

Then $\mu M dx + \mu N dy = 0$ is exact.

So $\exists F$ s.t.

$$(1) F_x = \mu M = x^2 - e^{-y} \Rightarrow F(x, y) = \frac{x^3}{3} - x e^{-y} + g(y) \quad \text{int. w/ } x$$

$$(2) F_y = \mu N = y^3 + x e^{-y} = x e^{-y} + g'(y) \quad \text{diff. w/ } y$$

$$\Rightarrow g'(y) = y^3 \Rightarrow g(y) = \frac{y^4}{4} + d$$

$$\text{So } F(x, y) = \frac{x^3}{3} - x e^{-y} + \frac{y^4}{4} + d = c_1$$

$$\boxed{\frac{x^3}{3} - x e^{-y} + \frac{y^4}{4} = c}$$

ex find the condition s.t $\mu = \mu(x, y)$ is an integrating factor for DE

$$M dx + N dy = 0 \quad \text{which is not exact}$$

Sch let $t = xy$, clearly $\mu_y = \frac{d\mu}{dt} t_y = \mu'_x$

$$\mu = \mu(xy) = \mu(t)$$

$$\mu(x) = \mu'_y$$

μ will be integr. factor if DE become exact after mult. with μ

$$(\mu M)_y = \mu_y M + \mu M_y$$

$$(\mu N)_x = \mu_x N + \mu N_x$$

$$\Rightarrow \mu'_x M + \mu M_y = \mu'_y N + \mu N_x$$

$$\rightarrow \frac{\mu'_x}{\mu} = \frac{N_x - M_y}{\mu_x N_y} \quad \begin{array}{l} \nearrow t \text{ only} \\ \nearrow \text{must be } t \text{ only} \end{array}$$

exerc Show $(x^3 y^4 + x^2 y^6) dx + (2x^2 y^5 - x^2 y^7) dy = 0$ is not exact

~~xxxx~~

and find an integrating factor $\mu = \mu(t)$, $t = xy^2$ solve DE

Modelling

1) (Tank Problem) A tank originally ^{contains} 100 liters of pure water. Then water containing $\frac{1}{2}$ kg of salt per liter is poured into tank at a rate of 2L/min, and mixture is allowed to leave at the same rate.

- a.) Write a DE with IC for $Q(t)$ = amount of salt in the tank at time t
 b.) The soln. of DE above is $Q(t) = 50(1 - e^{-t/50})$

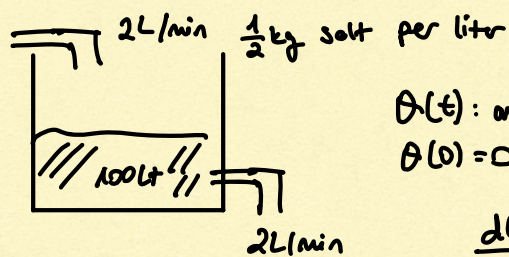
After 10 min the process is stopped, and pure water is poured into the tank at a rate of 2L/min with mixture leaving at the same rate. Write a new DE with IC for

$R(t)$ = amount of salt in tank at time $t \geq 10$.

2.) A tank with a capacity of 500L originally contains 200L of water with 100kg of salt in solution. Water containing 1 kg of salt per liter is entering at a rate of 3L/min and the mixture is allowed to leave tank at a rate of 2L/min.

- a.) Write a DE for $Q(t)$ before the time when the tank is full with IC.
 b.) Solve the DE and compute $Q(t)$ when tank is full
 c.) After the tank is full, it will overflow, so that water leaves the tank 3L/min (same as entering). Write a new DE for the change of $Q(t)$ after tank is full with IC.

Soln 1) Tank Problem



$Q(t)$: amount of salt at time t
 $Q(0) = 0$

$\frac{dQ}{dt}$ = rate of change of salt in tank by time

$$\frac{dQ}{dt} = \text{Rate In} - \text{Rate Out}$$

$$\frac{dQ}{dt} = 2 \cdot \frac{1}{2} - 2 \cdot (\text{concentration}) = 2 \cdot \frac{Q(t)}{100}$$

concentration of salt in the mixture
 $\frac{\text{amount of salt}}{\text{total amount of mixture}}$

total amount of mixture
 $Q(t) = 100$

Since the same amount of mixture is going in and going out the total amount inside tank does not change. So it is 100L

$$\frac{dQ}{dt} = 1 - \frac{Q(t)}{50}, \quad Q(0) = 0$$

$$\frac{dQ}{dt} = \frac{50 - Q(t)}{50} \rightarrow \int \frac{dQ}{50 - Q(t)} = \int \frac{dt}{50} \rightarrow \text{soln. } Q(t) = 50(1 - e^{-t/50})$$

Soln 1) b) (Tank Problem)

$$Q(t) = 50 (1 - e^{-t/50})$$

$$R(t) = Q(100) = 50(1 - e^{-1/5})$$

$$\frac{dR}{dt} = \frac{R(t)}{50}, \quad R(t) = 50(1 - e^{-t/50})$$

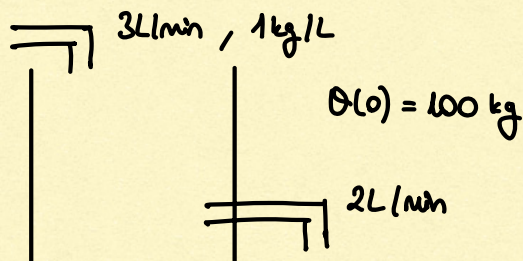
Rate In - Rate Out

$$\frac{dR}{dt} = 2L/min \cdot 0 kg/min - 2 \cdot \frac{R(t)}{100}$$

$$\frac{dR}{R} = \frac{-1}{50} dt \quad \ln|R| = \frac{-1}{50} t \quad R = c e^{\frac{-1}{50} t} \quad (t=0) \rightarrow R(0) = 50(1 - e^{-1/5}) = c e^{\frac{-1}{50} \cdot 0}$$

$$c = 50(1 - e^{-1/5}), \quad R = 50(1 - e^{-1/5}) e^{-t/50}$$

Soln 2)



Total amount of Mixture
 " $200 + t$
 " $(3t - 2t)$

$$\frac{dQ}{dt} = 3L/min \cdot 1kg/min - 2L/min \cdot \frac{Q(t)}{200+t}, \quad Q(0) = 100$$

Soln b) $\frac{dQ}{dt} + \frac{2}{200+t} Q(t) = 3$ (linear)

$$\mu = e^{\int P(t) dt} \rightarrow (200+t)^2 \Rightarrow [(200+t)^2 Q]' = 3 \cdot (200+t)^2$$

$$Q(t) = (200+t) - \frac{C}{(200+t)^2}, \quad Q(0) = 100 \rightarrow C = -4 \cdot 10^6$$

$$Q(t) = (200+t) - \frac{4 \cdot 10^6}{(200+t)^2}, \quad \text{tank is full when } 200+t = 500, \quad t = 300$$

$$Q(300) = 500 - \frac{4 \cdot 10^6}{25 \cdot 10^4} = 484$$

Soln c) $\frac{dR}{dt} = 3 \cdot 1 - \frac{3 \cdot R(t)}{500}, \quad R(t) = Q(300) = 484$ IVP

3.) (Newton's Second Law of Motion)

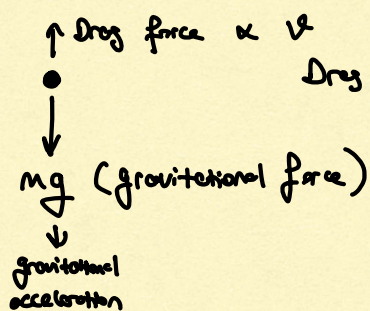
An object of mass m is dropped from rest in a medium which offers resistance proportional to the velocity v of the object. Assuming gravitational force to be constant find velocity and position of object at time t .

4.) (Newton's Law of cooling) Temperature of an object changes at a rate proportional to the difference in temperature between the medium and body itself

A thermometer, which shows 22°C inside a house, is placed outside where the temperature is 10°C . Two minutes later the thermometer shows 16°C . Predict the thermometer reading at any time

5.) In the absence of any factors, the population doubles every week. However, predator (birds, bats, ...) eat 20000 mosquitoes everyday. If there are 200000 mosquitoes initially, determine population of mos. in the area at any time.

Soln 3) (Newton 2nd Law of Motion)



Drag force $= -k v$ $\rightarrow k > 0$ k (coefficient)

net force $= \overset{\text{mass}}{m} \cdot \overset{\text{acceleration}}{a}$

$= mg - k \cdot v$

$a = \frac{dv}{dt}$

so $\rightarrow \boxed{\frac{dv}{dt} = g - \frac{k}{m} v}$ $\overset{\text{IVP}}{v(0)=0}$

Solve $\rightarrow v(t) = \frac{mg}{k} - \frac{mg}{k} \cdot e^{-\frac{kt}{m}}$
 $\leftarrow \begin{matrix} \frac{dv}{dt} = \frac{dx}{dt} \\ x(0)=0 \end{matrix}$ $t \rightarrow \infty$ soln $\frac{mg}{k}$

$v =$

12 Mar Pzt

Modelling

4. Newton's Law of Cooling

$S(t)$ = temperature of object at time t

A thermometer: $S(t)$

$S(0) = 22$

$\frac{ds}{dt} \propto (S(t) - T_m)$ [T_m : medium temperature]

$T_m = 10$ $S(2) = 16$

$\frac{ds}{dt} = -k(S(t) - T_m)$, $k > 0$
 $\downarrow$
 temp of object decreases.

find $S(t)$ at any time

IVP: $\frac{ds}{dt} = -k(S(t) - T_m)$, $S(0) = 22$ $S(2) = 16$
 $\downarrow$
 10

Solve DE

$$\int \frac{ds}{s(t) - 10} = \int -k dt$$

$$s(t) = 10 + ce^{-kt}$$

$$s(0) = 22 = 10 + c \quad c = 12$$

$$s(1) = 16 = 10 + 12e^{-k} \rightarrow k = \ln 2$$

$$s(t) = 10 + 12e^{-t \ln 2}$$

$$\int \frac{ds}{s - 10} = \int -k dt$$

$$\ln(s - 10) = -kt + C_1$$

$$s - 10 = e^{-kt + C_1}$$

$$s - 10 = e^{-kt} \cdot C$$

$$s = 10 + e^{-kt} \cdot C$$

$$C = 12$$

$$s = 10 + e^{-kt^2} \cdot 12$$

$$16 = 10 + 12e^{-2k}$$

$$\sqrt{2}$$

5. Population Dynamics

$\theta(t)$: population of mosquitoes after t days

$$\frac{d\theta}{dt} \propto \theta(t) \Rightarrow \frac{d\theta}{dt} = k \cdot \theta(t), \quad k > 0$$

$$\theta(0) = 200.000$$

$$\theta(7) = 400.000$$

, find $\theta(t)$

$$\int \frac{d\theta}{\theta} = \int k dt \rightarrow \ln \theta = kt + C_1$$

$$\theta(t) = e^{C_1 kt} = e^{C_1} \cdot e^{kt} = C \cdot e^{kt}$$

$$\theta(0) = 200.000$$

$$\theta(7) = 400.000 = 200.000 e^{k7} \rightarrow k = \frac{\ln 2}{7} \quad e^{kt} = e^{\frac{\ln 2}{7} t} = (e^{\ln 2})^{\frac{t}{7}} = 2^{\frac{t}{7}}$$

$$\text{So } \theta(t) = 200.000 \cdot 2^{\frac{t}{7}}$$

in the presence of predators

Predators eat 20.000 mosq per day, Model the situation

$$\frac{d\theta}{dt} = k \cdot \theta(t) - 20.000 \quad \text{"linear"}$$

$$\mu(t) = e^{\int -k dt} = e^{-kt} = 2^{-\frac{t}{7}}$$

$$\Rightarrow e^{-kt} \cdot \frac{d\theta}{dt} - k e^{-kt} \theta = -20.000 e^{-kt}$$

$$(e^{-kt} \cdot \theta)' \quad \text{then integrate}$$

$$\theta(t) = \frac{20.000}{k} + C e^{kt}$$

$$\text{put } k = \frac{\ln 2}{7} \text{ and } C, \theta(0) = 20.000$$

Approximate Soln to ODE

$$\text{VP} \begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Now suppose DE is not in one of known types: linear, sepr. etc

then, use numerical techniques to approx. the soln of DE

Picard's Iteration (Successive approx. method)

consider VP $y' = f(t, y)$, $y(t_0) = y_0$ satisfies the cond. in E and U Theorem

that is $f(t, y)$ is cont. on $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ and $\frac{\partial f}{\partial y}$ is also cont on R

that is,

$$n=0$$

i) choose an initial guess $y_0(t) = y(t_0) = y_0$ for all t

"Picard iterates"

ii) define $y_{n+1}(t) = y_0 + \int_{t_0}^t f(s, y_n(s)) ds$, $n = 0, 1, 2, \dots$

Thm: under the cond. of E and U Thm, the successive approximation (Picard iteration)

$$y_{n+1}(t) = y_0 + \int_{t_0}^t f(s, y_n(s)) ds, \quad y_0(t) = y_0$$

converges to unique soln of VP $y' = f(t, y)$, $y(t_0) = y_0$ in interval $[t_0 - h, t_0 + h]$

that is $\lim_{n \rightarrow \infty} y_{n+1}(t) = \lim_{n \rightarrow \infty} y_n(t) = \Phi(t)$ where $\Phi(t)$ is the soln to VP ($y_n(t) \approx \Phi(t)$)

thus $\exists h > 0$ st unique soln $y(t)$ exists on interval $[t_0 - h, t_0 + h]$

then the VP can be transformed into an equivalent integral eqn.

$$\int_{t_0}^t y'(s) ds = \int_{t_0}^t f(s, y(s)) ds = y(t) - y(t_0) = \int_{t_0}^t f(s, y(s)) ds$$

$$\Rightarrow y(t) = y(t_0) + \int_{t_0}^t f(s, y(s)) ds \quad \text{"Integral eqn"} \Rightarrow \boxed{y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds}$$

to solve this, we use following algorithm known as "Picard iteration" which aims to construct a sequence of functions $\{y_n\}$ that will converge to the desired soln of IVP.

ex $y' = y$, $y(0) = 1$ exact soln $y = e^t$

$$f(t, y) = y \quad \left\{ \begin{array}{l} \text{cont } v(t, y) \in \mathbb{R}^2 \\ E \text{ and } V \text{ cond. are} \\ \text{satisfied.} \end{array} \right.$$

$$\frac{\partial f}{\partial y} = 1$$

1) $y(0) = 1$

$$y_{n+1} = y_0 + \int_{t_0=0}^t \overbrace{f(s, y(s))}^{y(s)} ds = 1 + \int_0^t y_n(s) ds$$

$$n=0 \Rightarrow y_1 = 1 + \int_0^t f_0(s) ds = 1 + t$$

$$n=1 \Rightarrow y_2 = 1 + \int_0^t y_1(s) ds = 1 + t + \frac{t^2}{2}$$

$$n=2 \Rightarrow y_3 = 1 + \int_0^t y_2(s) ds = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

$$n=n-1 \Rightarrow y_n = 1 + \int_0^t y_{n-1}(s) ds = 1 + t + \dots + \frac{t^n}{n!}$$

$$y_n = \sum_{k=0}^n \frac{t^k}{k!}$$

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{t^k}{k!} = e^t$$

ex $y' = 2(y+1)$, $y(0) = 0$ solve by Picard iteration

$$y_0(t) = 0$$

$$y_{n+1}(t) = 0 + \int_0^t 2(y(s)+1) ds = 2 \int_0^t (y(s)+1) ds$$

$$n=0 \quad y_1(t) = 2 \int_0^t (y_0+1) ds = 2t$$

$$n=1 \quad y_2(t) = 2 \int_0^t \underbrace{(2s+1)}_{y_1} ds = \frac{(2t)^2}{2!} + 2t$$

$$n=2 \quad y_3(t) = \int_0^t \underbrace{\left(\frac{(2s)^2}{2!} + 2s+1 \right)}_{y_2} ds = y_3(t) = \frac{(2t)^3}{3!} + \frac{(2t)^2}{2!} + 2t \Rightarrow y_n(t) = \sum_{k=1}^n \frac{(2t)^k}{k!}$$

the soln will be

$\lim_{n \rightarrow \infty} y_n$ since $f = 2(y+1)$ and $\frac{\partial f}{\partial y}$ are cont everywhere

$$\sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = e^{2t} - 1 = y(t)$$

remar: Picard iteration is NOT useful when $f(t, y)$ is complex.

ex $y' = t^2 + y^2$, $y(0) = 0$

first $f(t, y) = t^2 + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are cont. everywhere ✓

$$y_0(t) = 0$$

$$y_{n+1}(t) = 0 + \int_0^t [s^2 + y_n^2(s)] ds$$

$$y_1(t) = \int_0^t [s^2 + 0^2] ds = \frac{t^3}{3}$$

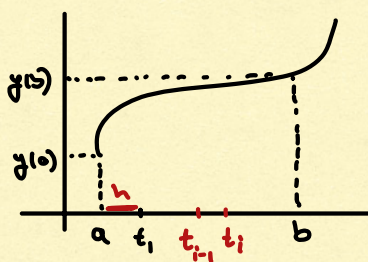
$$y_2(t) = \frac{t^3}{3} + \frac{t^7}{7 \cdot 9}$$

$$y_3(t) = \int_0^t [s^2 + \left(\frac{s^3}{3} + \frac{s^7}{7 \cdot 9}\right)^2] ds = ??$$

not easy at all to find y_1

Euler's Method

$$y' = f(t, y), \quad t \in [a, b], \quad y(a) = A, \quad y(b) = ?$$



$$t_i = a + ih$$

assumption: IVP has a soln on $[a, b]$ that is

it satisfies E and U Thm. . if $y(t)$ is a soln what is $y(b)$?

Discretize $[0, b]$ into n equal subintervals

$$A = y(a) = y(t_0)$$

$$\text{step size} = h = \frac{b-a}{n}, \quad t_n = t_0 + nh, \quad t_n = a + nh,$$

now we define Euler's Method by

$$y_{n+1} = y_n + h f(t_n, y_n)$$

$$y(t_0) = y(a) = A$$

approx sol $y_n \approx y(t_n)$ exact

or $y_{n+1} \approx y(t_{n+1})$

Derivation of Euler's Formula

$$y_{n+1} = y_n + h f(t_n, y_n)$$

1) From Calculus: lim defn of derivative

$$y'(t) = \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h}$$

$$y'(t) \approx \frac{y(t+h) - y(t)}{h} \text{ for sufficiently small } h$$

$$y'(t_n) \approx \frac{y(t_{n+1}) - y(t_n)}{h} \Rightarrow y(t_{n+1}) \approx y(t_n) + h y'(t_n) \stackrel{\text{VP}}{=} f(t_n, y(t_n))$$

Using Taylor Series Expansion

taylor series exp. of $y(t_n+h)$ around t_n

$$y(t_n+h) = y(t_n) + h y'(t_n) + \underbrace{\frac{h^2}{2} y''(\xi)}_{\text{error}} \dots \in [t_n, t_{n+1}]$$

$$y(t_{n+1}) \approx y(t_n) + h y'(t_n) \rightarrow f(t_n, y(t_n))$$

$\Downarrow$

$y_{n+1} = y_n + h \cdot f(t_n, y_n)$

$$|\text{Error}| = |\text{exact value} - \text{approx value}|$$

$$|e_n| = |y(t_n) - y_n| \leq O(h^2) \stackrel{||}{=} \frac{h^2}{2} y''(\xi)$$

order of h^2

$O(h^{1+p})$: p. order of the numerical method

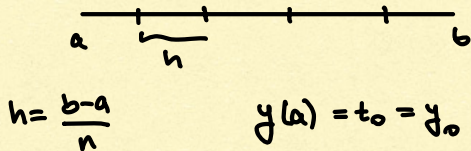
* order of Euler Method is 1

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Approximate Solns

$$\left. \begin{array}{l} y' = f(t, y) \\ y(t_0) = y_0 \end{array} \right\} \text{ IVP} \quad \text{Picard iteration: } y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds, \quad y_0(t) = y_0$$

Euler's Method: $y_{n+1} = y_n + h f(t_n, y_n)$, $y(t_0) = y_0$ (given)



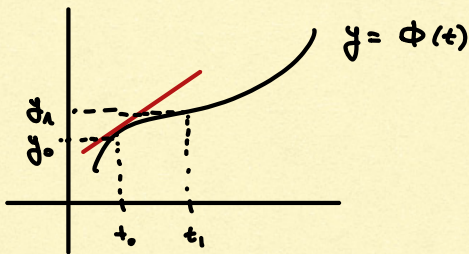
$h = \frac{b-a}{n}$ $y(a) = t_0 = y_0$

$\leftarrow y_n \approx y(t_n) \rightarrow$
approx exact

$$|\text{error}| = |\text{exact} - \text{approx}| = \left| \frac{h^2}{2} y''(\xi) \right| \in (t_n, t_{n+1}) \Rightarrow \text{let } |y''(\xi)| \leq M$$

$$\Rightarrow |\text{error}| \leq \frac{M}{2} h^2 \rightarrow o(h^1)$$

Geometric meaning of Euler Method



tangent line to one soln:

$$y - y_0 = m(t - t_0)$$

$$y = y_0 + (t - t_0) f(t_0, y_0) \quad \text{say } t = t_1$$

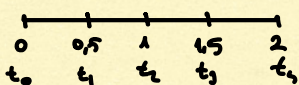
$$y(t_1) = y_0 + \underbrace{(t_1 - t_0)}_h f(t_0, y_0)$$

$$y_1 = y(t_1) = y_0 + h \cdot f(t_0, y_0)$$

$y_2 = y(t_2) = y_1 + h f(t_1, y_1)$ (to find y_2 draw tangent line to another soln of DE at (t_1, y_1))

ex $y' = t + y$, $y(0) = 1$, $y(2) = ?$ by Euler Method $h = 0.5$

Soln



$$y_{n+1} = y_n + h f(t_n, y_n)$$

$$|e_4| \leq \frac{M}{2} h^2 = \frac{M}{8} \quad \max \{y''(t)\} = M \quad t \in [a, b]$$

$$y_{n+1} = y_n + h(t_n + y_n) \rightarrow y_1 = y_0 + 0.5(t_0 + y_0) = 1.5$$

$$y_3 = y_2 + 0.5(t_2 + y_2) = 17/8$$

$$y_2 = y_1 + 0.5(t_1 + y_1) = 2.5$$

$$y_4 = y_3 + 0.5(t_3 + y_3) = 57/8$$

Analytically

$$y' - y = t \quad \mu = e^{\int -1 dt} = e^{-t} \Rightarrow (e^{-t} y)' = t e^{-t}$$

$$e^{-t} y = \int t e^{-t} dt \quad u=t, \quad e^{-t} dt = dv \quad du = dt \quad e^{-t} = v$$

$$e^{-t} y = -t e^{-t} + \int e^{-t} dt$$

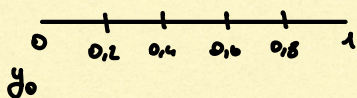
$$y = t - 1 + c e^t$$

$$y(0) = 1 = -1 + c \quad c = 2$$

$$y = t - 1 + 2e^t$$

ex $y' = y^2, \quad y(1) = ? \quad y(0) = \frac{1}{2} \quad \text{take } h=0.2 \rightarrow n=5$

Sch



$$y_{n+1} = y_n + h f(t_n, y_n)$$

$$y_{n+1} = y_n + h y_n^2$$

$$y_1 = 1.2 y_0^2 = 1.2 \cdot \left(\frac{1}{2}\right)^2 = 0.35, \quad y_2 = y_1 + h y_1^2 = 0.61$$

$$y_3 = 0.685 \quad y_4 = 0.77 \quad y_5 = 0.90 \dots \approx y(1)$$

exact sch

$$\int \frac{dy}{y^2} = \int dt \rightarrow -\frac{1}{y} = t + c$$

$$y = -\frac{1}{t+c} \quad y(0) = \frac{1}{2} \quad c = -2$$

$$y(t) = -\frac{1}{t-2}, \quad y(1) = 1$$

The Relatives of Euler's Method

the recurrence relation

↓ 2 method

$$y_{n+1} = y_n + h \left[(1-\theta) f(t_n, y_n) + \theta f(t_{n+1}, y(t_{n+1})) \right]$$

weighted avg of slope $= y' = f(t, y)$

at two successive points where θ is a parameter st $\theta \in [0, 1]$
 (t_0, y_0) and (t_{n+1}, y_{n+1})

Remark when $\theta = 0$ gives us Euler's Method

improved Euler's Thm.
 $\left\{ \theta = \frac{1}{2} \quad y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \right.$

$\theta = 1$ $y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$ = "Backward Euler Method"

Remark: improved Euler Method is a two-stage technique to calculate y_{n+1} , we need

to first predict a value for

$f(t_{n+1}, y_{n+1})$ sitting on RHS of improved Euler formula. To predict we can use Euler method.

ex $y' = \frac{1-t+4y}{t}$, $y(0)=1$

use improved euler $h=0.025$

$$f(t_n, y_n) = f_n = 1 - t_n + 4y_n$$

$$f(t_{n+1}, y_{n+1}) = f_{n+1} = 1 - t_{n+1} + 4y_{n+1} = y_{n+1} f_n$$

Predict by using Euler Method.

$$y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

" $\xrightarrow{\text{prediction}}$ $y_n + h f(t_n, y_n)$

$y_{n+1} = y_n + \frac{h}{2} [f_n + f(t_{n+1}, y_n + h f_n)]$

 $\Rightarrow$ improved euler

t_1	t_2
0	0.025
t_0	0.05

$y_0=1$ $y_1?$ $y_2?$

Predict $\tilde{y}_1 = y_0 + h f_0 = 1 + 0.025(1-0+4) = 1.125$

correct $y_1 + \frac{h}{2} [f(t_0, y_0) + f(t_1, \tilde{y}_1)]$

so $y_1 = 1.1309 \dots$

$$f_1 = 1 - 0.025 + 4(1.1309 \dots) = 5.49 \dots$$

$$y_1 + h f_1 = 1.1309 \dots + 0.025(5.49 \dots) = 1.268 \dots$$

$$f_2 = f(t_2, y_1 + h f_1) = 1 - 0.05 + 4(1.268 \dots) = 6.02 \dots$$

$$y_2 = 1.1309 \dots + 0.5 \cdot 0.025(5.49 \dots + 6.02 \dots) = 1.2743$$

$$y_3 = y_2 + \frac{h}{2} [f_2 + f(t_3, \tilde{y}_3)]$$

ex $y' = t - y^2$, $y(0) = 0$ $t \in [0, 0.4]$ $h = 0.1$

$t_0 = 0$ 0.1 0.2 0.3 $0.4 = t_4$
 $y_0 = 0$ $y_4 \approx y(0.4)$

n	t_n	y_n with $\theta=0$	$y_n, \theta = \frac{1}{2}$	$y_n, \theta=1$	exact
0	0	0	0	0	0
1	0.1	0	0.005	0.00999	0.005
2	0.2	0.01	0.01998	0.002930	0.01998
3	0.3	0.02999	0.04486	0.05955	0.04486
4	0.4	0.05990	0.07949	0.08857	0.07949

(note: improved euler method gives a result

with error $\leq 5 \cdot 10^{-4}$

but euler and backward euler gives result

with error = exact - approx $\leq 3 \cdot 10^{-2}$)

Differences between Linear and NonLinear DEs

1) the interval of existence of a soln of a non linear eqn cannot be predicted, but for

linear eqn it's known in advance by Existence and Uniqueness Thm.

ex $y' + \frac{2x}{4-x^2} y = \frac{3x^2}{4-x^2}$ $y(1) = -3$ $p(x), q(x)$ are cont if $x \neq \pm 2$

$I = (-2, 2)$, f and $\frac{\partial f}{\partial y} = 4xy$ are cont $\mathbb{R}^2$

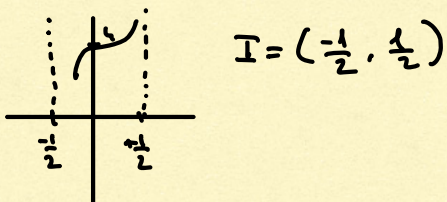


$\Rightarrow \exists$ a unique soln pass through $(0, y)$ $I = (-h, h)$, $h = ?$ $h > 0$? (we can't predict h !)

now solve the DE

$y' = 2xy^2$

$\frac{dy}{y^2} = 2x dx$ $-\frac{1}{y} = x^2 + c$ $y = -\frac{1}{x^2 + c}$ $c = -\frac{1}{c} \Rightarrow y = \frac{1}{x^2 - \frac{1}{c}}$ $x \neq \pm \frac{1}{\sqrt{c}}$



2) **general soln** is a soln containing all possible solns but we usually use the word general soln for linear eqns but NOT for nonlinear eqns.

ex $y' = y^2 \Rightarrow y = -\frac{1}{x+c} \rightarrow$ NOT A GENERAL SOLN

but $y(x) = 0$ is also another soln for DE but I can't obtain it from $y = -\frac{1}{x+c}$

Review of matrices

Defn: A matrix is an array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

i th row $\leftarrow$ $\uparrow$ # of rows
 $\rightarrow$ # of columns $\rightarrow$ size of matrix (m by n)
 $\rightarrow$ entry (element) of A, denoted by $A = [a_{ij}]$ or $A = (a_{ij})$
 $\downarrow$ j th column $\downarrow$ # of columns

Exm. $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -1 & 2 \end{bmatrix}_{2 \times 3}$ $a_{23} = 2$ $a_{12} = 2$

Defn: $A = (a_{ij})_{m \times n}$

• If $a_{ij} \in \mathbb{R} \Rightarrow A$ is called "Real Matrix".

$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ real matrix

• If $a_{ij} \in \mathbb{C}$: complex number $\Rightarrow A$ is called "complex matrix".

$A = \begin{bmatrix} i & 1+2i \\ -1 & i-2 \end{bmatrix}$ complex matrix

Complex numbers

$z = a + b\bar{i}$ $\rightarrow$ imaginary (z) $\left[i^2 = -1 \right]$ $\bar{z} = a - b\bar{i}$ conjugate of z .
 $\downarrow$ $\text{Re}(z)$: real part of z

• $z \bar{z} = (a + b\bar{i})(a - b\bar{i}) = a^2 - ab\bar{i} + ab\bar{i} - b^2 \overset{-1}{i^2} = a^2 + b^2 = |z|^2$

where $|z|$ is the magnitude or norm of z .

Properties of matrices:

$A = [a_{ij}]_{m \times n}$

1.) Transpose of A $\Rightarrow A^T = [a_{ji}]_{n \times m}$

$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}_{3 \times 2} \Rightarrow A^T = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}_{2 \times 3}$

2.) Complex conjugate of A: $\bar{A} = [\bar{a}_{ij}]$

$$A = \begin{bmatrix} i & 0 \\ -1 & 2-i \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 0 \\ -1 & 2+i \end{bmatrix}$$

3.) Adjoint of A: $A^* = \bar{A}^T$

$$A^* = \bar{A}^T = \begin{bmatrix} -i & 0 \\ -1 & 2+i \end{bmatrix}^T = \begin{bmatrix} -i & -1 \\ 0 & 2+i \end{bmatrix}$$

4.) Equality: $A = B \Rightarrow a_{ij} = b_{ij} \forall i, j$ (their size must be equal)

5.) Zero matrix $A = (a_{ij})$, $a_{ij} = 0 \forall i, j$. $A = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$

6.) Identity matrix $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}_{n \times n}$
 main diagonal
 $a_{ii} = 1$
 $a_{ij} = 0, i \neq j$
 $I_{n \times n} = I_n$

7.) A is a square matrix if #rows = #columns

8.) Addition $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ (A, B must have same size)

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$

$$A + B = B + A$$

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}_{2 \times 2}, \quad B = \begin{bmatrix} 0 & -2 \\ 2i & i \end{bmatrix}_{2 \times 2}$$

$$A + B = \begin{bmatrix} -1 & -2 \\ 2+2i & 3+i \end{bmatrix}_{2 \times 2}$$

9.) Multiplication by a scalar c:

$$cA = c(a_{ij}) = (ca_{ij})$$

$$A = \begin{bmatrix} i & 1 \\ 0 & -2 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2i & 2 \\ 0 & -4 \end{bmatrix}$$

10.) Multiplication of two matrices

$$A_{m \times n} B_{n \times p} = C_{m \times p}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$c_{11} = 1 \cdot 2 + 2(-1) + (-1) \cdot 0 \quad c_{12} \quad c_{21}$$

• $AB \neq BA$ (in general)

• $A(BC) = (AB)C$

• $A(B+C) = AB+AC$

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix}_{2 \times 3}$$

$$B = \begin{bmatrix} 2 & i \\ -1 & 0 \\ 0 & 3 \end{bmatrix}_{3 \times 2}$$

$$\Rightarrow AB = \begin{bmatrix} (1 \cdot 2 + 2(-1) + (-1) \cdot 0) & 1 \cdot i + 2 \cdot 0 + (-1) \cdot 3 \\ (0 \cdot 2 + 3(-1) + 1 \cdot 0) & 0 \cdot i + 3 \cdot 0 + 1 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i-3 \\ -3 & 3 \end{bmatrix}$$

$$A_{m \times n} B_{n \times p} = C_{m \times p} \quad AB$$

11. Multiplication of vectors

• $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$: column vector $n \times 1$

• $x = [x_1 \ x_2 \ \dots \ x_n]_{1 \times n}$
row vector

$x = [1, 2, 3]$

$y = [0, -1, 0]$

$x \cdot y = 1 \cdot 0 + 2(-1) + 3 \cdot 0 = -2$

Inner (dot / scalar) Product:

$x = [x_1 \ x_2 \ \dots \ x_n]$ • $x \cdot y = x y^T = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^n x_i y_i$
when x, y are real vectors

• If $x, y \in \mathbb{C}^{n \times 1} \Rightarrow x \cdot y = \sum_{i=1}^n x_i \bar{y}_i$ ↓ and $x \cdot y = y \cdot x$

• If $x \cdot y = 0 \Rightarrow x \perp y$ (they are perpendicular / orthogonal to each other)

Norm of x : $\|x\| = \sqrt{x \cdot x} \Rightarrow \bar{y} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$

Ex : $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 1-i \end{bmatrix}$ $x \cdot y = (x, y) = 1(-1) + i(1-i) = i-2$

12.) Inverse of A A is square, $A_{n \times n}$

$AB = BA = I$
 $B = A^{-1} \neq \left(\frac{1}{A}\right)$

A matrix $A_{n \times n}$ has an inverse if there is a matrix $B_{n \times n}$ s.t.

$AB = BA = I_n \rightarrow$ identity matrix

• Such a matrix A is called invertible (nonsingular). The matrix B is called inverse of A , and is denoted by $B = A^{-1}$

• If A has NO inverse, it is called singular or non-invertible.

Finding A^{-1} :

By using

1.) Gaussian Elimination: (Elementary row operations)

Elementary row operations:

a.) Interchanging two rows : $R_i \leftrightarrow R_j$ $A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$

b.) Multiplying a row with a constant c : cR_i $A \xrightarrow{2R_2} \begin{bmatrix} 1 & 2 \\ -6 & -2 \end{bmatrix}$

c.) Multiplying a row with a constant $c \neq 0$, and add it to another row:
 $\downarrow R_1$
 $cR_i + R_j$: $A \xrightarrow{2R_1 + R_2} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

To find A^{-1} : Write augmented matrix

$$\left[\begin{array}{c|c} A & I_{n \times n} \end{array} \right] \xrightarrow[\substack{\text{apply} \\ \text{row operations} \\ \text{to convert } A \text{ into } I}]{\substack{\downarrow \\ \text{identity}}} \left[I \mid A^{-1} \right]$$

Ex: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array} \right] \xrightarrow{3R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 7 & 3 & 1 \end{array} \right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/7 & 1/7 \end{array} \right]$

$\xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1/7 & -2/7 \\ 0 & 1 & 3/7 & 1/7 \end{array} \right]$ So $A^{-1} = \begin{bmatrix} 1/7 & -2/7 \\ 3/7 & 1/7 \end{bmatrix}$

$\downarrow$ I $\downarrow$ A^{-1}

2.) Using Determinants

Determinant $A_{n \times n}$: square

$n=1$, $A = [a_{11}] \Rightarrow \det A = |A| = a_{11}$

$n=2$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = |A| = a_{11}a_{22} - a_{12}a_{21}$

$n=3$, $A = \begin{bmatrix} a_{11}^+ & a_{12}^- & a_{13}^+ \\ a_{21}^- & a_{22}^+ & a_{23}^- \\ a_{31}^+ & a_{32}^- & a_{33}^+ \end{bmatrix} \Rightarrow \det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$(-1)^{i+j}$

$n \times n$: $\det A = \sum_{j=1}^n (-1)^{\bar{i}+j} a_{\bar{i}j} \det(A_{\bar{i}j})$

$\downarrow$
is the matrix obtained from A
by deleting $\bar{i}$ th row and j th column

Exm: $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix}$

$= -1 - 2(2-3) + 0 = 1$

Theorem let A be $n \times n$ matrix. Then A^{-1} exists iff $\det A \neq 0$.

Defn: $A^{-1} = \frac{1}{\det A} C^T$ = adjoint of A

Remark $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2 \times 2}$
 $\Rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $C_{ij} = [(-1)^{i+j} \det A_{ij}]$ with A_{ij} is the matrix obtained by deleting the i -th row and j -th column of the original matrix A .

Exm $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\det A = (-2-1) - (4-1) - (2+1) = -3-3-3 = -9 \neq 0$
 so A^{-1} exists.

$C_{11} = (-1)^{1+1} \det A_{11}$ where $A_{11} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} \Rightarrow C_{11} = -3$
 $\det A_{11} = -2-1 = -3$

$C_{12} = (-1)^{1+2} \det A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$

$C_{13} = (-1)^{1+3} \det A_{13} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$

$\vdots$

Thus, $C = \begin{bmatrix} -3 & -3 & 3 \\ -3 & 3 & 0 \\ 0 & -3 & 3 \end{bmatrix}$ with $C^T = \begin{bmatrix} -3 & -3 & 0 \\ -3 & 3 & -3 \\ 0 & 0 & 3 \end{bmatrix}$

Then, $A^{-1} = \frac{1}{\det A} C^T = \frac{1}{-9} \begin{bmatrix} -3 & -3 & 0 \\ -3 & 3 & -3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ -1/3 & 0 & -1/3 \end{bmatrix}$

Effects of row operations on $\det A$:

1) $c R_i \rightarrow R_i$ ($c \neq 0$) $\Rightarrow$ multiplies $\det A$ by c

2) $R_i \leftrightarrow R_j$ ($i \neq j$) $\Rightarrow$ multiplies $\det A$ by (-1)

3) $c R_i + R_j \rightarrow R_j$ $\Rightarrow$ does not change $\det A$.

Note: $\det(cA) = c^n \det A$
 if A is of size $n \times n$.

Exm: $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ $\det A = 3+2=5$

($\det B = 2 \cdot 4 - 3 \cdot 6 = -10$)

$\Uparrow$

$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 \\ 2 & -2 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} = B$
 $\det A = 5$
 $\det B = -2 \det A = -10$

13. Matrix Functions

matrices of which entries are functions.

$$A = \begin{bmatrix} e^{3t} & t \ln t & 1 \\ 2t & e^{2t} & \cos t \end{bmatrix}$$

$$\frac{dA}{dt} = \frac{d[a_{ij}(t)]}{dt} = \begin{bmatrix} 3e^{3t} & \ln t + t \cdot \frac{1}{t} & 0 \\ 2 & 2e^{2t} & -\sin t \end{bmatrix}$$

$$\int A dt = \left[\int a_{ij}(t) dt \right] = \begin{bmatrix} \frac{e^{3t}}{3} & t \ln t + t & t \\ t^2 & \frac{e^{2t}}{2} & \sin t \end{bmatrix}$$

if $a_{ij}(t)$ are differentiable and integrable

14. Triangular Matrices:

- **Diagonal matrix** (main diagonal) $\begin{bmatrix} * & & 0 \\ & * & \\ 0 & & * \end{bmatrix}$ $a_{ii} \neq 0$ for some i
 $a_{ij} = 0, i \neq j$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- **Upper Triangular matrix** $\begin{bmatrix} * & * & * \\ & * & * \\ 0 & & * \end{bmatrix}$ $a_{ij} = 0, i > j$ $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$
- **Lower Triangular matrix** $\begin{bmatrix} * & & 0 \\ & * & \\ * & & * \end{bmatrix}$ $a_{ij} = 0, i < j$ $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

System of Linear Algebraic Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

"System of linear eqns. in n unknowns"
 m -equations
 n -unknowns
 $x_1, x_2, \dots, x_n$ are unknowns

This can be written as $\downarrow$ unknown vector

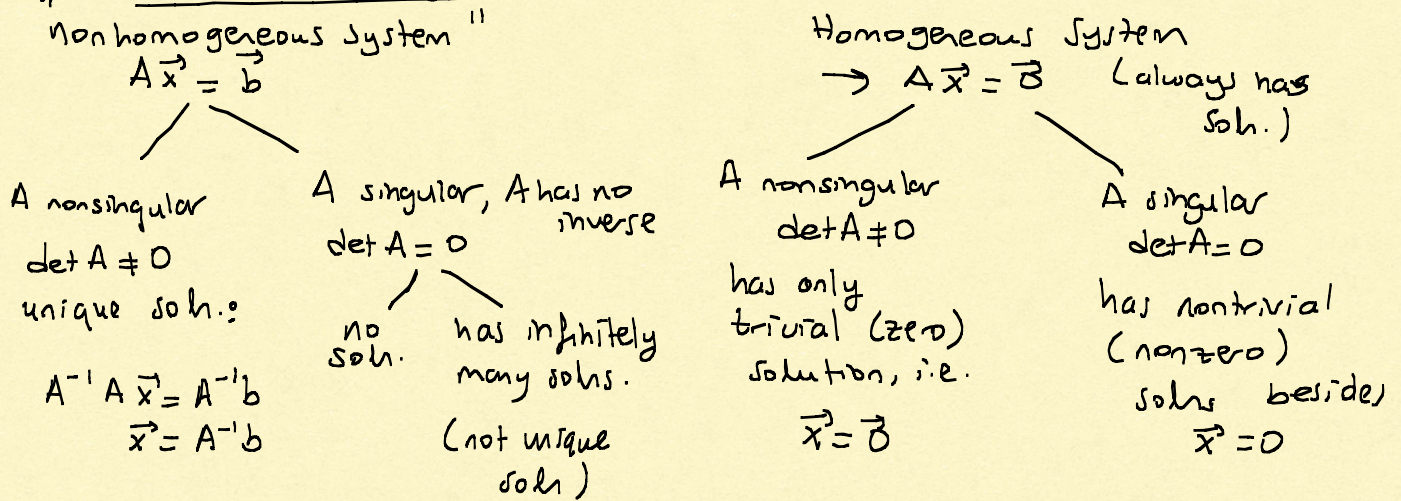
$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A: \text{coefficient matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}} \Rightarrow \boxed{A\vec{x} = \vec{b}}$$

right hand side vector, given

$$\begin{aligned} \underbrace{A^{-1}A}_{I}x &= A^{-1}b \\ x &= A^{-1}b \\ \vec{x} &= \vec{0} \end{aligned}$$

Defn If $b=0$, then $A\vec{x}=\vec{0}$ and it is called homogeneous, otherwise, $A\vec{x}=\vec{b}$ is called non-homogeneous.

" For square systems, $A_{n \times n}$, we have followings:



* The soln. of the system $A\vec{x}=\vec{b}$ can be found by using

Gaussian elimination. writing the augmented matrix

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & & & b_2 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array} \right] \xrightarrow[\text{operations}]{\text{apply row}} \left[\begin{array}{ccc|c} \text{upper triangular matrix} & & & \vec{b} \end{array} \right]$$

coefficient matrix A
RTS vector

use backward substitution to obtain soln.

Exm

$$\begin{aligned} x_1 - x_3 + x_4 &= 0 \\ 2x_1 + x_2 + x_3 &= 1 \\ -x_1 + x_2 + 2x_3 - x_4 &= 2 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ -1 & 1 & 2 & -1 & 2 \end{array} \right] \xrightarrow[R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 1 & 1 & 0 & 2 \end{array} \right] \xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & -3 & 3 & 1 \end{array} \right]$$

$$\xrightarrow{-1/3 R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -3 & 1 \\ 0 & 0 & 1 & -1 & -1/3 \end{array} \right]$$

leading 1's (leading variables) (the first nonzero entry of each row is called leading variable).

The columns that do not contain a leading 1 correspond to free variables.

Here x_4 : free variable

x_1, x_2, x_3 are leading unknowns. This means that x_1, x_2, x_3 can be written in terms of x_4 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 - x_4 \\ x_4 - 1/3 \\ x_4 \end{bmatrix}$$

Then, the last eqn. will be .

$$\begin{aligned} x_1 - x_3 + x_4 &= 0 \Rightarrow x_1 = x_3 - x_4 = x_4 - \frac{1}{3} - x_4 = -\frac{1}{3} \\ x_2 + 4x_3 - 3x_4 &= 1 \Rightarrow x_2 = 1 + 3x_4 - 4x_3 = 1 + 3x_4 - 4x_4 + 4/3 = 7/3 - x_4 \\ \Rightarrow x_3 - x_4 &= -1/3 \Rightarrow \boxed{x_3 = x_4 - 1/3} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 - x_4 \\ x_4 - 1/3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 \\ -1/3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

So we have infinitely many solns.

Defn: # of leading entries (1's) is called the rank of matrix A.

Thus, in previous example $\text{rank}(A) = 3$

Exm

$$\begin{aligned} x_1 - x_2 + 3x_3 + x_4 + x_5 &= -1 \\ x_1 - 2x_3 + x_4 + 2x_5 &= 3 \\ 2x_1 - 2x_2 + 6x_3 + 4x_5 &= 2 \end{aligned}$$

$$\begin{aligned} \downarrow & \quad \quad \quad \downarrow \quad \downarrow \\ \left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 1 & 0 & -2 & 1 & 2 & 3 \\ 2 & -2 & 6 & 0 & 4 & 2 \end{array} \right] & \xrightarrow[-2R_1+R_3]{-R_1+R_2} & \left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 0 & 1 & -5 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 2 & 4 \end{array} \right] & \xrightarrow{-1/2 R_3} & \left[\begin{array}{ccccc|c} 1 & -1 & 3 & 1 & 1 & -1 \\ 0 & 1 & -5 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{array} \right] \end{aligned}$$

x_1, x_2, x_4 : leading 1's $\Rightarrow \text{Rank } A = 3$ so x_1, x_2, x_4 can be written in terms of x_3, x_5 .

$$\begin{aligned} x_1 - x_2 + 3x_3 + x_4 + x_5 &= -1 \Rightarrow x_1 = -1 + x_2 - 3x_3 - x_4 - x_5 = 5 + 2x_3 - 3x_5 \\ x_2 - 5x_3 + x_5 &= 4 \uparrow \Rightarrow x_2 = 4 + 5x_3 - x_5 \\ x_4 - x_5 &= -2 \Rightarrow x_4 = -2 + x_5 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 + 2x_3 - 3x_5 \\ 4 + 5x_3 - x_5 \\ x_3 \\ -2 + x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

"infinitely many solns."

Remark In Gaussian elimination, we aim to convert to given matrix A into a new matrix, which is called row reduced echelon form of the A .

Defn: A matrix satisfying the following properties is called row reduced echelon form:

- 1.) The first non-zero entry of each row is 1, which are called leading 1.
- 2.) All entries which are below a leading 1 are 0.
- 3.) If $i < j$ then the leading 1 on row i must be to the left of the leading 1 on row j .
- 4.) If there are k rows of 0's in the matrix, they must be last k rows.

Exm

$$\begin{bmatrix} \textcircled{1} & 0 & 1 & 2 \\ 0 & \textcircled{1} & 3 & -5 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

leading 1's
"row-reduced echelon"

$$\begin{bmatrix} \textcircled{1} & -5 \\ 0 & \textcircled{1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

leading 1's
"row-reduced echelon"

$$\begin{bmatrix} \textcircled{3} & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{not row-reduced echelon}$$

$$\begin{bmatrix} 0 & \textcircled{1} & -1 \\ \textcircled{1} & 0 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & -1 \end{bmatrix}$$

Thm Any matrix can be reduced until a matrix in row echelon form is obtained.

Exm: 1.) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{1/3 R_1} \begin{bmatrix} \textcircled{1} & 2/3 & -1/3 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$

leading 1's, no free variable
row-reduced echelon

2.) $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & -1 \end{bmatrix}$ "row-reduced echelon"

Leading 1's. $\rightarrow$ one free variable

Linear Independence

Defn If $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$ be vectors, then

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \dots + c_n \vec{x}^{(n)}$$

is called the linear combination of $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$ for some nonzero constants $c_i, i=1, 2, \dots, n$.

Exm $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{x}^{(2)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow 2\vec{x}^{(1)} - 3\vec{x}^{(2)}$ is a linear combination of $\vec{x}^{(1)}, \vec{x}^{(2)}$.

Defn: A set of n vector $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$ is called

"linearly independent" if

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \dots + c_n \vec{x}^{(n)} = \vec{0}$$

$$\begin{aligned} \vec{x}^{(1)} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \vec{x}^{(2)} &= \begin{bmatrix} -3 \\ -6 \end{bmatrix} \\ \vec{x}^{(2)} &= -3\vec{x}^{(1)} & \vec{x}^{(2)} &= \begin{bmatrix} 0 \\ 3 \end{bmatrix} \end{aligned}$$

implies that $c_1 = c_2 = \dots = c_n = 0$. Other-wise, it is called

"linearly dependent." In other words, a set of vectors is linearly

independent iff none of the vectors can be expressed as a linear combination of remaining vectors.

let $\vec{x}^{(i)} = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix}$, then $c_1 \vec{x}^{(1)} + \dots + c_n \vec{x}^{(n)} = \vec{0}$ will give us the following linear homogeneous system:

$$c_1 \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} + c_2 \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix} + \dots + c_n \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_n^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

unknown

$$\Rightarrow \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \underline{X} \vec{c} = \vec{0}$$

"homogeneous linear system" with unknown c_i .

Conclusions 1.) If $\det X \neq 0 \Rightarrow X$ is invertible and the system has only the trivial (zero) solution. That is,
 $\vec{c} = \vec{0}$ (i.e. $c_i = 0 \quad \forall i = 1, \dots, n$).

Thus, the vectors are linearly independent.

2.) If $\det X = 0 \Rightarrow$ system has non-zero solutions. That is
 $c_i \neq 0$ for some i . Thus, the vectors $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$
 are linearly dependent.

Remark Thus, $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$ are linearly independent iff $\det X \neq 0$.

Exm Determine the vectors $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{z} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$ are linearly independent or not.

Construct the system $[\vec{x} \ \vec{y} \ \vec{z}] \vec{c} = \vec{0}$ s.t.

$$\underbrace{\begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix}}_X \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{"solve this hom. system"}$$

$$\det X = (-11-3) - 2(-22+1) - 4(6+1) = -14 + 42 - 28 = 0 \Rightarrow \text{system has}$$

non-zero solutions, that is, some $c_i \neq 0$. Thus, the vectors are linearly dependent, that is, vectors can be written as linear

combinations of others. Find these combinations by solving the system.

$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 \\ \uparrow & \uparrow & \uparrow \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array} \right] \xrightarrow[\substack{-2R_1 + R_2 \\ R_1 + R_3}]{\substack{R_1 + R_3 \\ -2R_1 + R_2}} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{array} \right] \xrightarrow[\substack{1/5 R_3}]{-1/3 R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array} \right] \xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$\downarrow$
 this vector, no need to write

leading ones
 c_3 is free variable.

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0 \Rightarrow c_1 = -2c_2 + 4c_3 \\ c_2 - 3c_3 &= 0 \Rightarrow c_2 = 3c_3 \end{aligned}$$

(so c_1, c_2 can be written in terms of c_3 .)

Then nontrivial soln: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

let $c_3 = 1 \Rightarrow$
 $\boxed{-2\vec{x}^{(1)} + 3\vec{x}^{(2)} + \vec{x}^{(3)} = \vec{0}}$

Exm $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ Check linear independence. $c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)} = \vec{0}$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\underline{X}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First way

(1) $\det \underline{X} = 1 + 1 - 1 = 1 \neq 0$ so system has only trivial (zero) sol. $\vec{c} = \vec{0}$

Thus, $c_1 = c_2 = c_3 = 0 \Rightarrow$ vectors are linearly independent.

Second way

(2) Solve the system by Gaussian elimination (long way)

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{-1/3 R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} c_1 - c_2 - c_3 = 0 \\ c_2 + 2c_3 = 0 \\ c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = c_3 = 0$$

So vectors are linearly independent.

Defn A set of functions $\{f_1(t), f_2(t), \dots, f_n(t)\}$ is linearly independent if the equation

$$c_1 f_1(t) + c_2 f_2(t) + \dots + c_n f_n(t) = 0 \quad \text{for all } t$$

imply $c_1 = c_2 = \dots = c_n = 0$.

Exm Show that $e^{r_1 t}$ and $e^{r_2 t}$, $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$ are linearly independent.

$c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0$ for all t . Choose some specific values of t to obtain a linear system.

$$\begin{aligned} t=0 &\Rightarrow c_1 + c_2 = 0 \\ t=1 &\Rightarrow c_1 e^{r_1} + c_2 e^{r_2} = 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 \\ e^{r_1} & e^{r_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\det \underline{X} = e^{r_2} - e^{r_1}}_{\neq 0 \text{ since } r_2 \neq r_1} \neq 0$$

So, $c_1 = c_2 = 0 \Rightarrow e^{r_1 t}$ and $e^{r_2 t}$ are linearly independent.

Remark There is another way to check linear independence of functions $\{f_1(t), \dots, f_n(t)\}$ if $f_i(t)$ are differentiable enough times.

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \quad \text{holds for all } t.$$

Differentiate this eqn. repeatedly in order to construct a linear system:

$$\begin{cases} n\text{-eqns.} \\ n\text{-unknowns} \end{cases} \begin{cases} c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \\ c_1 f_1' + c_2 f_2' + \dots + c_n f_n' = 0 \\ c_1 f_1'' + c_2 f_2'' + \dots + c_n f_n'' = 0 \\ \vdots \\ c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \dots + c_n f_n^{(n-1)} = 0 \end{cases} \Rightarrow$$

$$\underbrace{\begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{bmatrix}}_F \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

"hom. linear system"

$\det F \neq 0 \Rightarrow F$ is invertible and $c_1 = c_2 = \dots = c_n = 0$.

Thus, f_i are linearly independent functions.

Defn $\det F = W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$

is called the Wronskian of $f_1, f_2, \dots, f_n$.

Theorem If the Wronskian of a set of functions is NOT zero even only for one value of t , then the set is linearly independent.

In other words, $W(f_1, f_2, \dots, f_n) \neq 0$ for some $t \Rightarrow$

$f_1, f_2, \dots, f_n$ are linearly independent.

Exm $f_1 = e^{r_1 t}, f_2 = e^{r_2 t}, r_1 \neq r_2 \in \mathbb{R}$

$$W(f_1, f_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{(r_1 + r_2)t} \underbrace{(r_2 - r_1)}_{\neq 0 \text{ since } r_1 \neq r_2} \neq 0 \quad \forall t \Rightarrow f_1, f_2 \text{ are linearly independent.}$$

Exm $f_1 = \cos t$, $f_2 = \sin t$

$$W(f_1, f_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0 \quad \forall t \text{ so}$$

$\{\cos t, \sin t\}$ are linearly independent.

Exm $f_1 = 1$, $f_2 = x$, $f_3 = x^2$

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0 \text{ so they are linearly independent.}$$

Eigenvalues and Eigenvectors

The linear system

$$A \vec{v} = \lambda \vec{v}$$

$$\rightarrow \lambda \in \mathbb{C}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{v}$$
$$A \vec{v} = \lambda \cdot \vec{I} \vec{v}$$

where both $\vec{v} \neq \vec{0}$ and scalar λ are unknown, is called an eigenvalue problem. λ is called "eigenvalue of A" and $\vec{v} \neq \vec{0}$ is a "corresponding eigenvector."

Rewrite the system as:

I: identity

$$A \vec{v} = \lambda \vec{v} \Rightarrow$$

$$A \vec{v} - \lambda \vec{v} = \vec{0}$$

$$B = (A - \lambda I) \vec{v} = \vec{0}$$

for $\vec{v} \neq \vec{0}$
soln:

$$(A - \lambda I) \vec{v}$$

"hom. linear system" has non-zero solns. if coefficient matrix is singular ($\det(A - \lambda I) = 0$), and we look for non-zero solns, i.e. $\vec{v} \neq \vec{0}$.

" $\det(A - \lambda I) = 0$ " gives an n^{th} degree polynomial in λ as:

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

"roots of this polynomial are the eigenvalues of A".

Once λ is obtained, one can find the corresponding eigenvector by solving the system

$$(A - \lambda I) \vec{v} = \vec{0} \text{ by inserting the obtained } \lambda \text{ value.}$$

Remark Clearly if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A second degree polynomial

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)} \lambda + \underbrace{ad-bc}_{\det A} = 0$$

Exm Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$

Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) - 0 = \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda_1 = -3 \quad \lambda_2 = -1$$

$$\lambda_1 = -3: (A - \lambda_1 I) \vec{v} = \vec{0}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$(A + 3I) \vec{v} = \vec{0} \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} v_1: \text{leading } 1 \\ v_2: \text{free} \end{array}$$

$$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

wlog let's choose $v_2 = 1$

$\Rightarrow \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = -3$.

Soln space for $(A + 3I) \vec{v} = \vec{0}$ = spanned $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$\lambda_2 = -1 \quad (A - \lambda_2 I) \vec{u} = \vec{0} \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})$$

$$(A + I) \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} u_2: \text{leading} \\ u_1: \text{free} \end{array}$$

$$u_2 = 0, \quad u_1: \text{free}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

let wlog $u_1 = 1 \Rightarrow \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenvector corresponding to $\lambda = -1$

eigenvector = span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Note here that eigenvectors are not unique.

Defn: A matrix A is called diagonalizable if there exists an invertible matrix P s.t.

$$J = P^{-1} A P \quad \text{is a diagonal matrix}$$

$$\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

where $\lambda_i, i=1, \dots, n$ are eigenvalues of $A_{n \times n}$ and

$$P = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_n], \quad \text{where } \vec{v}_i, i=1, \dots, n \text{ are the}$$

eigenvectors corresponding to eigenvalues $\lambda_i, i=1, \dots, n$.

or

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$\downarrow$

$$J = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Exm (Previous Exm) A is diagonalizable since $\exists P = [\vec{v} \quad \vec{u}] = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow J = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = P^{-1} A P \quad \text{check (exercise)}$$

$$A \rightarrow J$$

Exm $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Find eigenvalues and corresponding eigenvectors.

exercise

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \stackrel{\uparrow}{=} -(\lambda+1)^2(\lambda-2) = 0$$

$\lambda_{1,2} = -1$ repeated eigenvalue / algebraic multiplicity of $\lambda = -1$
 $\lambda_3 = 2$
 $\downarrow$
 $\text{alg}(\lambda=2) = 1$ $\text{alg}(\lambda=-1) = 2$

$\lambda_3 = 2$ $(A - 2I)\vec{v} = 0$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 + R_3]{\frac{1}{2}R_1 + R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \xrightarrow[-2/3 R_2]{R_2 + R_3} \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

v_1, v_2 : leading
 v_3 : free

$$-2v_1 + v_2 + v_3 = 0 \Rightarrow v_1 = \frac{1}{2}(v_2 + v_3) = v_3$$

$$v_2 - v_3 = 0 \Rightarrow v_2 = v_3$$

$$\Rightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Soln space of $(A - 2I)\vec{v} = 0$ = $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = V$
 $\dim V = 1 = \# \text{ of free variables}$

$\lambda = -1$: $(A + I)\vec{u} = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[-R_1 + R_3]{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

u_1 : leading 1
 u_2, u_3 : free

$$u_1 + u_2 + u_3 = 0 \Rightarrow u_1 = -u_2 - u_3$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_2 - u_3 \\ u_2 \\ u_3 \end{bmatrix} = u_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Soln space of $(A + I)\vec{v} = \vec{0}$ = spanned $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim V = 2 = \text{alg}(\lambda = -1)$
 $\parallel$ $\parallel$ $\parallel$
 V $\vec{u}(1)$ $\vec{u}(2)$ $\# \text{ of free variables}$
 $\parallel$
 geometric multiplicity of $\lambda = -1$

Since $m(\lambda = -1) = \text{alg}(\lambda = -1)$, then A is diagonalizable. That is $\exists$ an invertible matrix P , $m(\lambda = -1) = 2$

$$P = [\vec{v} \quad \vec{v}^{(1)} \quad \vec{v}^{(2)}] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \det P \neq 0. \text{ and}$$

$$J = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exm $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$ Find eigenvalues and eigenvectors of A .

$\det(A - \lambda I) = 0$ " characteristic eqn (polynomial)

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = (2-\lambda)^3 = 0 \Rightarrow \lambda = 2, \text{ alg}(\lambda=2) = 3$$

$\lambda = 2: (A - 2I) \vec{v} = 0 \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} \textcircled{1} & -1 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

v_1, v_2 : leading 1's
 v_3 : free

1 free = 1 linearly independent eigenvector

$\therefore A$ is not diagonalizable.

$v_1 = v_2 + v_3 \Rightarrow v_1 = 0$
 $v_2 = -v_3$

$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ -v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow V = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\dim V = 1$
free variables
lin. indep. eigenvector

Two missing eigenvector (which can not be found by usual way of solving $(A - \lambda I) \vec{v} = 0$)
(esp. lin indep. too)

These are called generalized eigenvectors. The way to obtain them will be given later, when we explain Jordan forms.

Defn: given $\det(A - \lambda I) = (\lambda - \lambda_1)^{\alpha_1} \cdot (\lambda - \lambda_2)^{\alpha_2} \cdot \dots \cdot (\lambda - \lambda_n)^{\alpha_n}$, $\alpha_i = \text{alg}(\lambda_i)$

1) if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1 \Rightarrow$ no repeated eigenvalue that is λ_i are simple with algebraic multiplicity $= \text{alg}(\lambda_i) = 1$

if $\exists \alpha_i$ st $\alpha_i > 1$ then λ_i is a repeated eigenvalue with $\text{alg}(\lambda_i) = \alpha_i$

2) for a repeated eigenvalue with alg. mult. $\alpha_i > 1$, $\exists m_i \leq \alpha_i$ linearly independent eigenvectors. Here $m_i = \#$ of lin. independent eigenvectors is called geometric multiplicity

3) if $m(\lambda_i) = \alpha_i = \text{alg}(\lambda_i)$ for all i the matrix A is diagonalizable.

Systems of 1st order ODE

Defn the system of eqns in the following form

$$x_1' = f(t, x_1, \dots, x_n)$$

$$x_2' = f(t, x_1, \dots, x_n)$$

$\vdots$

$$x_n' = f(t, x_1, \dots, x_n)$$

x_i : dependent unknown variable

t : independent variable

ex $\frac{dx}{dt} = \overbrace{x_1 x_2 - 2x_1 t}^{f_1(t, x_1, x_2)}$ not linear

$$\frac{dx_2}{dt} = \overbrace{x_1 - x_2 t^2}^{f_2(t, x_1, x_2)}$$

$$\text{Thm: } \left. \begin{array}{l} x_1' = f_1(t, x_1, \dots, x_n) \\ x_2' = f_2(t, x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(t, x_1, \dots, x_n) \end{array} \right\} \text{①} \quad \text{with IC's} \quad \left. \begin{array}{l} x_1(t_0) = x_1^0 \\ x_2(t_0) = x_2^0 \\ \vdots \\ x_n(t_0) = x_n^0 \end{array} \right\} \text{②}$$

① + ② called an IVP if $f_1, f_2, \dots, f_n$ and their partial derivatives $\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_n}{\partial x_n}$ are continuous in a region R containing I . Point $(t_0, x_1^0, x_2^0, \dots, x_n^0)$ then $\exists$ an interval $|t - t_0| < h$ in which $\exists$ a unique soln for IVP. ① + ②

Defn: the IVP ① + ② has a soln if $\exists$ a set of differentiable functions $\{\phi_1(t), \phi_2(t), \dots, \phi_n(t)\}$ satisfy the DE ① + ②

Defn a system of 1st order ODE is **LINEAR** if it can be written in the form:

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + g_1(t)$$

$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + g_2(t)$$

$$\vdots$$

$$x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + g_n(t)$$

then we can write this with matrix vector form

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix} \Rightarrow \vec{x}' = A(t)\vec{x} + \vec{g}(t)$$

where $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$

Defn if $g_i(t) = 0 \quad \forall i = 1, \dots, n$ ($\vec{g}(t) = \vec{0}$) then the system is called

homogenous Other wise that is $\vec{g}(t) \neq \vec{0}$ some of $g_i(t) \neq 0$ then system is called non-homogenous

(N) $\vec{x}' = A(t)\vec{x} + \vec{g}(t)$ "non-homg"

(H) $\vec{x}' = A(t)\vec{x}$

ex $x_1' = 3x_1 - tx_2 + 1$
 $x_2' = tx_1 - x_2$

$$\vec{x}' = \underbrace{\begin{pmatrix} 3 & -t \\ t & -1 \end{pmatrix}}_{A(t)} \vec{x} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{g}(t)}$$

$f_1 = 3x_1 - tx_2 + 1$ $\frac{\partial f_1}{\partial x_1} = 3$
 $f_2 = tx_1 - x_2$ $\frac{\partial f_2}{\partial x_1} = t$

all cont $\forall (t, x_1, x_2)$

interval of soln $\forall t \in \mathbb{R}$ for any initial value (t_0, x_1^0, x_2^0)

Existence and Uniqueness for Linear System of 1st order ODE

if $A(t)$ and $\vec{g}(t)$ are cont. on I (interval) then IVP

$$x'(t) = A(t)x + \vec{g}(t)$$

has a unique soln defined $\forall t \in I$

$$x(t_0) = \vec{x}^0 \text{ (given)}$$

ex $x_1' = 3x_1 - tx_2 + \frac{1}{t^2-1}$

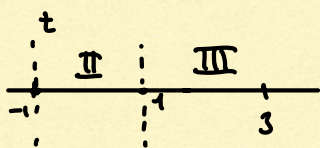
$$x'(3) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$x_2' = tx_1 - x_2$$

$$\vec{x}' = \underbrace{\begin{pmatrix} 3 & -t \\ t & -1 \end{pmatrix}}_{A(t)} \vec{x} + \underbrace{\begin{pmatrix} \frac{1}{t^2-1} \\ 0 \end{pmatrix}}_{\vec{g}(t)}$$

$\downarrow$
 cont $\forall t \in \mathbb{R}$

$\downarrow$
 cont $\forall t \in \mathbb{R} - \{-1, 1\}$



IVP has a unique soln on $t \in (1, \infty)$

remark: first consider the homogeneous system

$$\vec{x}' = A(t)\vec{x}$$

Thm: if $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are solns to

$$(H) \vec{x}' = A(t)\vec{x} \text{ it follows that their linear combination}$$

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n \text{ is also soln of (H) for any constant } c_i \text{ } i=1,2,\dots$$

(superposition principle)

Pf: $\vec{x}_1, \vec{x}_2$ are solns to $\vec{x}' = A\vec{x}$

$$c_1 / \vec{x}_1' = A(t) \vec{x}_1$$

$$c_2 / \vec{x}_2' = A(t) \vec{x}_2$$

$$c_1 \vec{x}_1' + c_2 \vec{x}_2' = c_1 \cdot A(t) \vec{x}_1 + c_2 A(t) \vec{x}_2$$

$$(c_1 \vec{x}_1 + c_2 \vec{x}_2)' = A(t) (c_1 \vec{x}_1 + c_2 \vec{x}_2) \Rightarrow c_1 \vec{x}_1 + c_2 \vec{x}_2 \text{ is a soln to } H$$

Defn a set of solns $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ of (H) are called linearly independent on I

if $c_1 \vec{x}_1 + \dots + c_n \vec{x}_n = 0 \quad \forall t \in I$ implies that $c_1 = c_2 = \dots = c_n = 0$

otherwise it's linearly dependent

Defn: a linearly independent set S of solns of (H) st every soln of (H)

can be written as a linear combination of elements of S is called a basis

for the space of solutions of (H) . Thus the set

$$S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\} \text{ is a basis for } (H)$$

Thm: if $\vec{x}_1, \dots, \vec{x}_n$ are lin indep. solns of system

$$(H) \quad \vec{x}' = A(t) \vec{x} \quad \text{for each } t \in I \quad \text{then each soln } (\vec{x} = \phi(t) \text{ of } (H))$$

can be written as a linear combination

$$\vec{x} = \phi(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n \quad \text{in a unique way and}$$

a general soln of (H)

Thm: Let $S = \{ \vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t) \}$ be a set of $\vec{x}'(t) = A(t) \vec{x}(t)$

defined on I and $\underline{X}(t) = [\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)]$

Then the followings are equivalent:

① S is linearly independent

② S is fundamental set of solns of H

③ $\underline{X}(t)$ is a fundamental matrix

④ $\Rightarrow$ ⑤

④ $\det \underline{X}(t) \neq 0 \quad \forall t \in I, \quad W(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$

⑤ $\det \underline{X}(t) = W(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \neq 0$ for some $t \in I$

⑥ $\vec{x}(t) = \underline{X}(t) \vec{c} = c_1 \vec{x}_1 + \dots + c_n \vec{x}_n$ is a general soln of H

⑦ $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$ is a general soln of $\vec{x}' = A(t) \vec{x} + \vec{g}(t) \quad \left(\begin{smallmatrix} N \\ \downarrow \\ \text{non homog} \end{smallmatrix} \right)$
 where $\vec{x}_h$: homog soln $\vec{x}_p$: partic soln

Defn A lin. indep. set of solns $\{ \vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t) \}$ of $\vec{x}'(t) = A(t) \vec{x}$ $\left(\begin{smallmatrix} H \\ \uparrow \\ \text{homog} \end{smallmatrix} \right)$ is called fundamental set of solns

The matrix $\underline{X}(t) = [\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)]$

which takes $\vec{x}_i$ (lin. indep. solns) as columns is called fundamental matrix

Properties

1) $\det \underline{X}(t) \neq 0 \quad \forall t \in I$ since $x_i(t)$ are lin. independent

and $\det \underline{X}(t) = W(\vec{x}_1, \dots, \vec{x}_n)$ called Wronskian of $\vec{x}_1 \dots \vec{x}_n$

2) $\underline{X}'(t) = [\underbrace{\vec{x}_1'(t)}_{A(t)\vec{x}_1}, \underbrace{\vec{x}_2'(t)}_{A(t)\vec{x}_2}, \dots, \underbrace{\vec{x}_n'(t)}_{A(t)\vec{x}_n}]$

$\underline{X}'(t) = A(t) [\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)]$
 $\underline{X}(t)$

$$\underline{X}'(t) = A(t) \cdot \underline{X}(t)$$

that is the fundamental matrix also satisfies the (H) $\vec{X}' = A(t) \vec{x}$

$$3) \underline{X}(t) \vec{c}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$\underline{X}(t) \vec{c} = [\vec{x}_1(t), \dots, \vec{x}_n(t)] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \dots + c_n \vec{x}_n(t)$$

that is $\underline{X}(t) \vec{c}$ is a general soln of (H)

$$\text{ex } \underbrace{\vec{x}' = \begin{pmatrix} -1 & 6 \\ 1 & -2 \end{pmatrix} \vec{x}}_{(H)} \quad I = (-\infty, \infty) \text{ interval of sol since } A(t) \text{ cont all}$$

$$\text{suppose } \vec{x}_1(t) = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{x}_2(t) = e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

show $\vec{x}_1, \vec{x}_2$ are soln to H

$$\vec{x}_1'(t) = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{so} \quad A \vec{x}_1(t) = \begin{pmatrix} -1 & 6 \\ 1 & -2 \end{pmatrix} e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{also } \vec{x}_2 \text{ as well}$$

is $\{\vec{x}_1(t), \vec{x}_2(t)\}$ linearly independent?

$$\text{check } W(\vec{x}_1, \vec{x}_2) = |\vec{x}_1 \vec{x}_2| = \begin{vmatrix} 3e^t & -2e^{-4t} \\ e^t & e^{-4t} \end{vmatrix} = 3e^{-3t} + 2e^{-3t} = 5e^{-3t} \neq 0 \forall t \in \mathbb{R} \\ \Rightarrow \text{lin. independent}$$

then $S = \{\vec{x}_1, \vec{x}_2\}$ is a fun. set of schs.

$$\text{then } \underline{X}(t) = [\vec{x}_1, \vec{x}_2] = \begin{pmatrix} 3e^t & -2e^{-4t} \\ e^t & e^{-4t} \end{pmatrix} \quad \text{is the fundamental matrix}$$

$$\text{then general soln } \vec{x}(t) = \underline{X}(t) \vec{c} = c_1 \begin{pmatrix} 3e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-4t} \\ e^{-4t} \end{pmatrix}$$

$$\text{find soln of } \vec{x}' = \begin{bmatrix} -1 & 6 \\ 1 & -2 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{by above general soln } \vec{x}(t) = c_1 e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{find } c_1, c_2 \text{ use } \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$t=0 \quad \vec{x}(0) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\text{so } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \rightarrow \begin{cases} 3c_1 - 2c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \quad \begin{matrix} c_2 = 3/5 \\ c_1 = 2/5 \end{matrix}$$

$$\boxed{\vec{x}(t) = \frac{2}{5} e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{3}{5} e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}} \\ \text{Soln to IVP}$$

how to find solns of 1st order linear system of DDEs

we'll consider the case A is constant

$$(N) \quad \vec{x}'(t) = A(t)\vec{x} + \vec{g}(t) \quad \text{non homo lin}$$

$$(H) \quad \vec{x}'(t) = A(t)\vec{x} \quad \text{homo lin.}$$

when A is constant

$$(\tilde{N}) \quad \vec{x}' = A\vec{x} + \vec{g}(t)$$

$$(\tilde{H}) \quad \vec{x}' = A\vec{x}$$

Note: only one DDE $x' = ax \Rightarrow \frac{dx}{x} = a dt \Rightarrow \ln x = at + c_1 \Rightarrow x = e^{c_1} e^{at} \Rightarrow x = C \cdot e^{at}$

then $x' = Ax$ $(\tilde{H})$ thus we may look for a soln of $(\tilde{H})$ in the form

$$\vec{x} = e^{\lambda t} \vec{v} \quad \vec{v} \neq \vec{0} \text{ constant}$$

similar to $(x = C e^{at})$

if $\vec{x} = e^{\lambda t} \vec{v}$ is a soln it must satisfy $(\tilde{H})$

substitute $\vec{x}$ in $(\tilde{H})$

$$x' = Ax$$

$$\lambda e^{\lambda t} \vec{v} = A e^{\lambda t} \vec{v}$$

$A\vec{v} = \lambda\vec{v}$

 (find eigenvalues)

λ : eigenvalues of A

$\vec{v} \neq \vec{0}$: corresponding eigenvectors

$$(A - \lambda I)\vec{v} = \vec{0} \quad (\text{homo, algb. system of eqns})$$

then for non zero solns $\det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$ (char. eqn)

for a poly. we have 3 cases for the roots

1) λ_i are real distinct

2) λ_i are complex conjugate

3) λ_i are repeated

Remark: if λ is eigenvalue of A with a corresponding eigenvector $\vec{v} \neq \vec{0}$

then $e^{\lambda t} \vec{v}$ is a soln of $\vec{x}' = A\vec{x}$

Case 1: Distinct Real Eigenvalue

$\lambda_1, \lambda_2, \dots, \lambda_n$ are real and distinct (not repeating) then each corresponds

to an eigenvector $\vec{v}_i \neq \vec{0}$ s.t. they are lin. independent. $i=1, \dots, n$

then $\{e^{\lambda_1 t} \vec{v}_1, e^{\lambda_2 t} \vec{v}_2, \dots, e^{\lambda_n t} \vec{v}_n\}$ is the set of fund soln of $(\vec{x}) \vec{x}' = A\vec{x}$

ex $x'_1 = -x_1 + 2x_2$
 $x'_2 = -3x_2$

$\vec{x}' = \overbrace{\begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}}^{A: \text{constant}} \vec{x}$

$\lambda_1 = -1 \quad \lambda_2 = -3$ distinct, real

$\text{alg}(\lambda_1) = 1$
 $\text{m}(\lambda_1)$

$\text{alg}(\lambda_2) = 1$
 $\text{m}(\lambda_2)$

$\lambda = -1 \rightarrow \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ free: # lin indep eigenvectors
leading

$v_2 = 0 \quad \vec{v}_1 = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Soln for eigenvector of $\lambda = -1$

$V_{\lambda=-1} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

so soln of system $e^{\lambda t} \vec{v}_1 = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -3: (A + 3I) \vec{v}_2 = \vec{0} \rightarrow \vec{v}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

v_1 leading
 v_2 free

$v_1 + v_2 = 0$
 $v_1 = -v_2$

$\vec{v}_2 = \begin{pmatrix} -v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so $V_{\lambda=-3} = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

the second soln: $e^{\lambda_2 t} \vec{v}_2 = e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

general soln: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

fund. matrix $\mathbf{X}(t) = [\vec{x}_1 \quad \vec{x}_2] = \begin{pmatrix} e^{-t} & e^{-3t} \\ 0 & e^{-3t} \end{pmatrix}$

remark: observe A is diagonalizable that is $\exists$ an invertible matrix P

$$D = P^{-1} A P$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} = P^{-1} A \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$A^\wedge$: since A is diagonalizable

$$D = P^{-1} A P \Leftrightarrow A = P D P^{-1}$$

$$A^\wedge = A \cdot A \cdot A \cdot \dots \cdot A = \underbrace{P D P^{-1} P D P^{-1} P D P^{-1}} \rightarrow A^\wedge = P D^\wedge P^{-1}$$

$$\text{where } D^\wedge = \begin{pmatrix} \lambda_1^\wedge & 0 \\ 0 & \lambda_2^\wedge \end{pmatrix}$$

$$\text{so } A^{10} = P \cdot \begin{pmatrix} (-1)^{10} & 0 \\ 0 & (-3)^{10} \end{pmatrix} P^{-1}$$

(in 2x2 we can draw the soln PHASE PORTRAIT)

$$A^{10} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

how to sketch the soln on x_1, x_2 system (Phase Portrait)

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad \vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (t \rightarrow \infty) \text{ take limit}$$

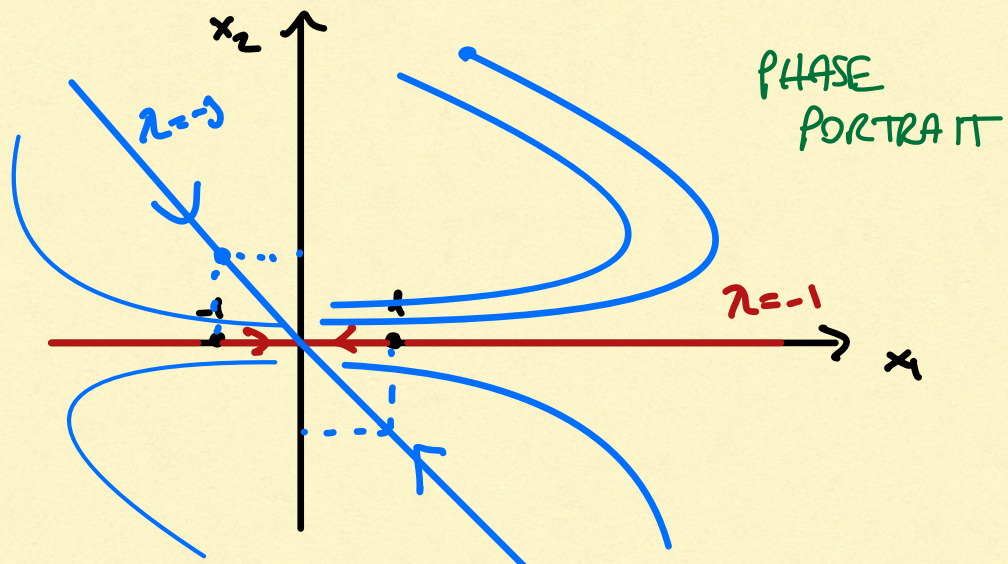
$$\begin{array}{ll} t \rightarrow \infty \rightarrow 0 & t \rightarrow \infty \text{ which one dominant } (e^{-t}) \text{ dominant} \\ t \rightarrow -\infty \rightarrow \infty & t \rightarrow -\infty \text{ " " " } (e^{-3t}) \text{ dominant} \end{array}$$

1) draw 4 easy solns

$$\begin{array}{ll} c_1 = 1 & c_2 = 0 \\ c_1 = 0 & c_2 = 1 \end{array}$$

equilibrium pt? $(0,0)$

origin is called node which is stable



Note: the sign of eigenvalue determine the direction of arrows in each soln curves

if $\lambda < 0$ $e^{\lambda t} \rightarrow 0$ as $t \rightarrow \infty$ arrow will be towards to origin "node stable" "sink"

if $\lambda > 0$ $e^{\lambda t} \rightarrow \infty$ as $t \rightarrow \infty$ arrow will be far away to origin "unstable source node"

27 Mart 2024

Case 1, real-distinct eigenvalues

ex $\vec{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix} \vec{x}$ Solve, draw phase portrait

eigenvalues

$$\det(A - \lambda I) \Rightarrow (2 - \lambda)(4 - \lambda) = 0 \quad \lambda_1 = 2 \quad \lambda_2 = 4$$

eigenvectors

$\lambda = 2 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $v_1 = \text{leading}$ $v_2 = \text{free}$ $v = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $V_{\lambda=2} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$
first soln $e^{\lambda_1 t} \rightarrow e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 4 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $v_1 = \text{leading}$ $v_2 = \text{free}$ $v = v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $V_{\lambda=4} = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

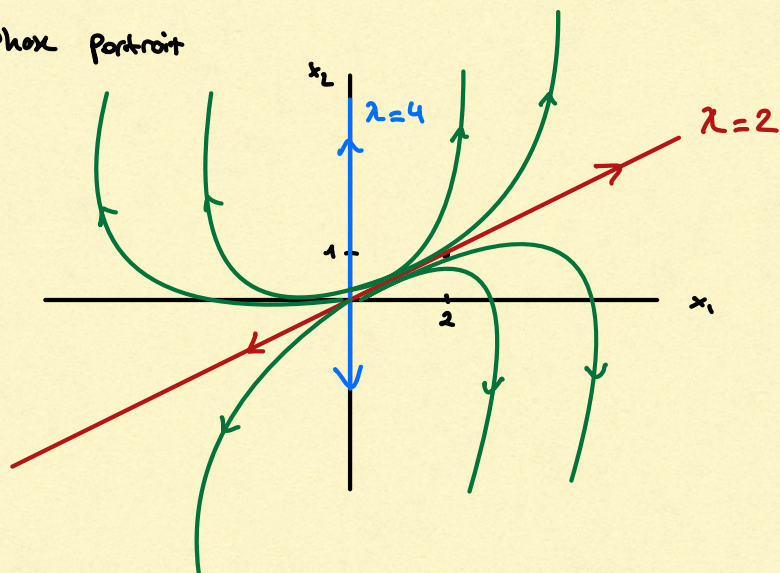
second soln $e^{\lambda_2 t} \rightarrow e^{4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x}_g = \underbrace{c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{dominant } t \rightarrow -\infty} + \underbrace{c_2 e^{4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{dominant } t \rightarrow \infty}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Phase Portrait



unstable
node

(node source)

ex

$$\vec{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix} \vec{x}$$

find soln, draw phase portrait

$$\lambda_1 = 2 \quad \lambda_2 = -4 \quad (1 \text{ positive, } 1 \text{ negative})$$

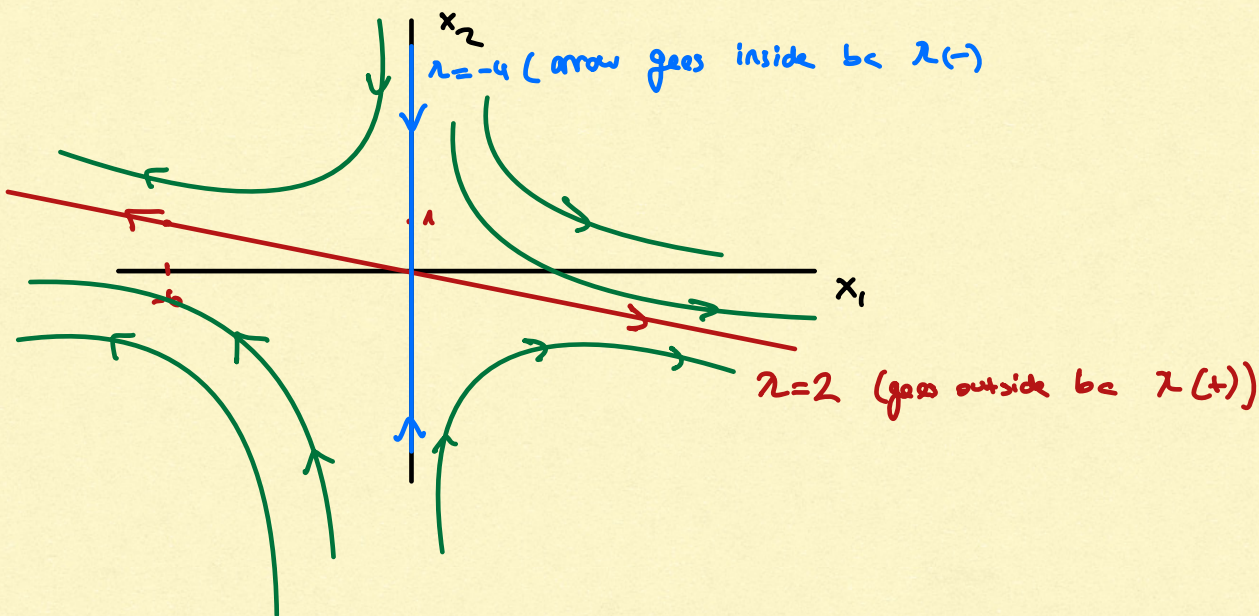
$$\lambda = 2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 \rightarrow \text{leading} \\ v_2 \rightarrow \text{free} \end{matrix}$$

$$v = v_2 \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

$$\lambda = -4 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} v_1 \rightarrow \text{leading} \\ v_2 \rightarrow \text{free} \end{matrix}$$

$$u = u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (u_1 = 0)$$

$$\vec{x}_g = \underbrace{c_1 e^{2t} \begin{pmatrix} -6 \\ 1 \end{pmatrix}}_{\substack{\text{dominant } t \rightarrow \infty \\ x_g \rightarrow \infty}} + \underbrace{c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\substack{\text{dominant } t \rightarrow -\infty \\ x_g \rightarrow \infty}}$$



ORIGIN IS
SADDLE POINT

UNSTABLE

ex $\vec{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \vec{x}$

$$\det(A - \lambda I) = (1-\lambda)(\lambda+1)(\lambda-4) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 4$$

$\lambda_1 = 1$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

v_1, v_2 leading

v_3 free

$$V_{\lambda=1} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$v_1 + v_2 + v_3 = 0 \rightarrow v_1 = v_3$$

$$\vec{x}_1(t) = e^{\lambda t} \vec{v} = e^t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$

$$A + I \vec{v} = \vec{0}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 5/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

v_1, v_2 lead

v_3 free

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\dim V_{\lambda=-1} = 1$$

$$V_{\lambda=-1} = \text{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\vec{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 4$

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$w_1 - 2w_2 + w_3 = 0$$

$$w_1 = w_3$$

$$w_2 - w_3 = 0 \rightarrow w_2 = w_3$$

$$\rightarrow w_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_3(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{x}_g = c_1 e^t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

fundamental matrix $\rightarrow \begin{bmatrix} e^t & -e^t & e^{4t} \\ -2e^t & 0 & e^{4t} \\ e^t & e^{-t} & e^{4t} \end{bmatrix}$

Case II Complex eigenvalues $\left(\begin{array}{l} \lambda = \alpha + i\beta, i^2 = -1 \\ \alpha, \beta \in \mathbb{R} \end{array} \right)$

Let $\lambda = \alpha + i\beta$ is an eigenvalue $\text{Re}(\lambda) = \alpha, \text{Im}(\lambda) = \beta$
with eigenvector $\vec{w} = \vec{u} + i\vec{v}$, $\vec{u}, \vec{v}$ real vectors.

Then:

$$A\vec{w} = \lambda\vec{w} \text{ take conjugate}$$

$$\bar{A}\bar{\vec{w}} = \bar{\lambda}\bar{\vec{w}}$$

Since A is real constant, then $\bar{A} = A$

$$\alpha - i\beta \quad A\bar{\vec{w}} = \bar{\lambda}\bar{\vec{w}}$$

So $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{\vec{w}} = \vec{u} - i\vec{v}$

Then, $\vec{x}_1 = e^{\lambda t} \vec{w}$ and $\vec{x}_2 = e^{\bar{\lambda} t} \bar{\vec{w}}$
are two linearly ind. solns. for $\vec{x}' = A\vec{x}$, which are complex solns.

So $\vec{x}_g = c_1 e^{\lambda t} \vec{w} + c_2 e^{\bar{\lambda} t} \bar{\vec{w}}$
is a complex soln

Remark In this course, we consider only real solns.

For real solns take suitable combination of $\vec{x}_1$ and $\vec{x}_2 = \bar{\vec{x}}_1$.

$$\frac{z + \bar{z}}{2} = \text{Re}(z)$$

$$\frac{z - \bar{z}}{2i} = \text{Im}(z)$$

$$c_1 = c_2 = \frac{1}{2} \left\{ \begin{array}{l} \phi_1 = \vec{x}_g = \frac{1}{2} \vec{x}_1 + \frac{1}{2} \vec{x}_2 = \frac{1}{2} (\vec{x}_1 + \bar{\vec{x}}_1) = \text{Re}(\vec{x}_1) \\ \phi_2 = \vec{x}_g = \frac{1}{2} \vec{x}_1 - \frac{1}{2} \vec{x}_2 = \frac{1}{2} (\vec{x}_1 - \bar{\vec{x}}_1) = \text{Im}(\vec{x}_1) = \text{Im}(e^{\lambda t} \vec{w}) \end{array} \right.$$

$$= \text{Re}(e^{\lambda t} \vec{w}) = \text{Re}(e^{(\alpha + i\beta)t} (\vec{u} + i\vec{v}))$$

$$c_1 = \frac{1}{2i}, c_2 = -\frac{1}{2i} \left\{ \begin{array}{l} \phi_1 = \vec{x}_g = \frac{1}{2i} (\vec{x}_1 - \bar{\vec{x}}_1) = \frac{1}{2i} (\vec{x}_1 - \bar{\vec{x}}_1) = \text{Im}(\vec{x}_1) = \text{Im}(e^{\lambda t} \vec{w}) \\ \phi_2 = \vec{x}_g = \frac{1}{2i} (\vec{x}_1 + \bar{\vec{x}}_1) = \frac{1}{2i} (\vec{x}_1 + \bar{\vec{x}}_1) = \text{Re}(\vec{x}_1) = \text{Re}(e^{\lambda t} \vec{w}) \end{array} \right.$$

where $\phi_1(t)$ and $\phi_2(t)$ are two real solns.

Then

$$\vec{x}_g = c_1 \phi_1(t) + c_2 \phi_2(t) = c_1 \text{Re}(\vec{x}_1) + c_2 \text{Im}(\vec{x}_1)$$

Now look at

$$e^{\lambda t} \vec{w} = e^{(\alpha + i\beta)t} (\vec{u} + i\vec{v}) = e^{\alpha t} (e^{i\beta t}) (\vec{u} + i\vec{v})$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

"Euler's Formula"

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$e^{\lambda t} \vec{w} = e^{\alpha t} (\cos \beta t + i \sin \beta t) (\vec{u} + i\vec{v})$$

$$= \underbrace{e^{\alpha t} (\cos \beta t \vec{u} - \sin \beta t \vec{v})}_{\text{Re}(e^{\lambda t} \vec{w}) = \text{Re}(\vec{x}_1(t))} + \underbrace{i e^{\alpha t} (\sin \beta t \vec{u} + \cos \beta t \vec{v})}_{\text{Im}(e^{\lambda t} \vec{w}) = \text{Im}(\vec{x}_1(t))}$$

$$\text{Re}(e^{\lambda t} \vec{w}) = \text{Re}(\vec{x}_1(t))$$

$$\text{Im}(e^{\lambda t} \vec{w}) = \text{Im}(\vec{x}_1(t))$$

$$\text{so } \vec{x}_g = c_1 \text{Re}(e^{\lambda t} \vec{w}) + c_2 \text{Im}(e^{\lambda t} \vec{w})$$

$$\vec{x}_g = c_1 e^{\alpha t} (\cos(\beta t) \vec{u} - \sin(\beta t) \vec{v}) + c_2 e^{\alpha t} (\sin(\beta t) \vec{u} + \cos(\beta t) \vec{v})$$

$$\alpha = \text{Re}(\lambda), \quad \beta = \text{Im}(\lambda) \quad \vec{u} = \text{Re}(\vec{w}) \quad \vec{v} = \text{Im}(\vec{w})$$

spiral grafik igari ni desori m?

Rename our real soln

$$\phi_1 = \vec{x}_1 = e^{\alpha t} (\cos(\beta t) \vec{u} - \sin(\beta t) \vec{v})$$

$$\phi_2 = \vec{x}_2 = e^{\alpha t} (\sin(\beta t) \vec{u} + \cos(\beta t) \vec{v})$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{x}_1 = e^{\alpha t} \cdot \begin{bmatrix} \cos \beta t u_1 - \sin \beta t v_1 \\ \sin \beta t u_1 + \cos \beta t v_1 \end{bmatrix}$$

$$\vec{x}_2 = e^{\alpha t} \begin{bmatrix} \sin \beta t u_1 + \cos \beta t v_1 \\ \sin \beta t u_2 + \cos \beta t v_2 \end{bmatrix}$$

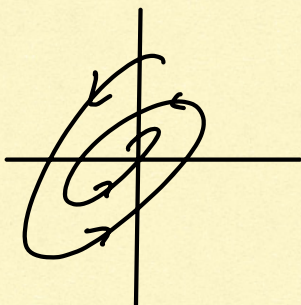
$$\begin{aligned} x_1^2 + x_2^2 &= e^{2\alpha t} \left[\cos^2 \beta t u_1^2 + \sin^2 \beta t v_1^2 - 2 \cos \beta t \sin \beta t u_1 v_1 \right. \\ &\quad \left. + \cos^2 \beta t u_2^2 + \sin^2 \beta t v_2^2 - 2 \cos \beta t \sin \beta t u_2 v_2 \right] \\ &\quad + e^{2\alpha t} \left[\sin^2 \beta t u_1^2 + \cos^2 \beta t v_1^2 + 2 \sin \beta t \cos \beta t u_1 v_1 \right. \\ &\quad \left. + \sin^2 \beta t u_2^2 + \cos^2 \beta t v_2^2 + 2 \sin \beta t \cos \beta t u_2 v_2 \right] \end{aligned}$$

$$x_1^2 + x_2^2 = e^{2\alpha t} \begin{bmatrix} u_1^2 + v_1^2 \\ u_2^2 + v_2^2 \end{bmatrix} \quad \text{if } \vec{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

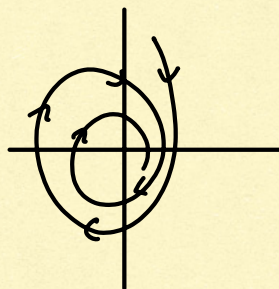
$$\begin{aligned} \Rightarrow x_1^2 + y_1^2 &= e^{2\alpha t} (u_1^2 + v_1^2) \\ x_2^2 + y_2^2 &= e^{2\alpha t} (u_2^2 + v_2^2) \end{aligned} \quad \left. \vphantom{\begin{aligned} \Rightarrow x_1^2 + y_1^2 &= e^{2\alpha t} (u_1^2 + v_1^2) \\ x_2^2 + y_2^2 &= e^{2\alpha t} (u_2^2 + v_2^2) \end{aligned}} \right\} \text{ellipse at } (0,0) \quad \text{if } \alpha \neq 0 \text{ with increasing } \alpha > 0 \\ &\quad \text{or decreasing } \alpha < 0$$

Phase Portrait for $\alpha = \operatorname{Re}(\lambda) < 0$

STABLE



counter clockwise



clockwise

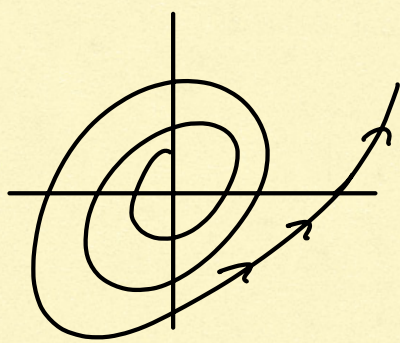
Spiral point
STABLE

Since $\alpha < 0$ all soln $\rightarrow$ origin

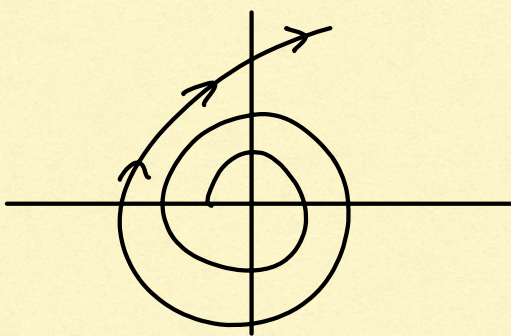
phase portrait for

$$\alpha = \operatorname{Re}(\lambda) > 0$$

UNSTABLE



counter clockwise



clockwise

phase portrait for

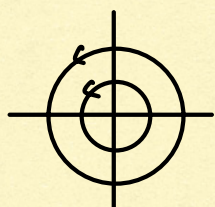
$$\alpha = \operatorname{Re}(\lambda) = 0$$

NEUTRALLY STABLE

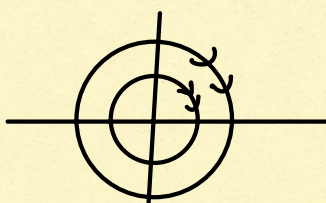
origin at center

and it is

NEUTRALLY STABLE



counter
clockwise



clockwise