Euclid's Elements Postulates

1) a straight line may be drawn from any one pt to any other pt.

axiom: given two distinct pts, there is a unique line that passes through them.

2) a terminated line can be produced indefinitely. Today we call line-segment is what Euclid called terminated line. So in todays terms, 2nd postulate says that a line segment can be extended on either side to form a line.

3) a circle can be drawn with any centre and any radius

OR given any straight line segment, a circle can be drawn having the segment as radius and one end point as center.

4) all right angles are equal to one another (congruent)

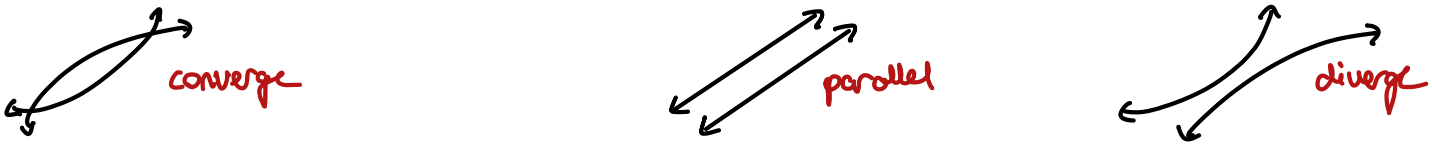
5) [parallel postulate] if two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles then the two lines inevitably must intersect each other on that side if extended far enough.

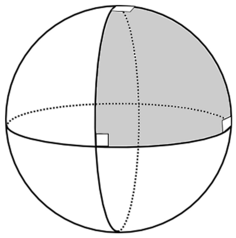
note unfortunately it's impossible to map a sphere to a plane without introducing distortion. for hyperbolic geometry we have "gnomonic, stereographic, orthographic

note in flat geometry we have parallel lines but in spherical lines they DNE any pair of straight lines will always converge and intersect eventually.

Q? dünyanın paralleri birbiri ile çakışmıyor ?
A: paraleller dünyanın spherical geometrisinin projeksiyonunun bir parçası değil.
bakman gereken lineler "great circles" ex: beach ball lines

SPHERICAL GEOMETRY   EUCLIDIAN GEOMETRY   HYPERBOLIC GEOMETRY



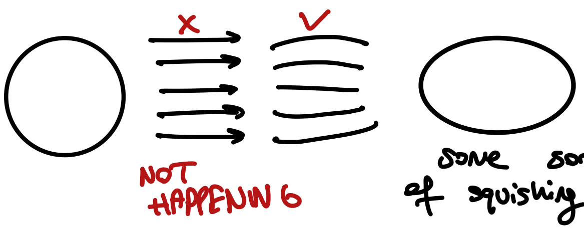


Spherical triangle üç açısı 90° üçgende gittik.

when we come back to the starting pt everything has been rotated 90° left
HOLONOMY
↳ DNE in Euclidian Geometry

Holonomy is opposite in hyperbolic plane

in spherical geometry when u move an object (objects are made up of smaller particles) all of its particles move as well. but remember there is no parallel lines in spherical geometry



similar to spaghettification around black hole caused by curved space

in hyperbolic space objects experience a stretching tidal force unlike spherical space which has stretching force.

Circumference of a circle

spherical geometry : $C = 2\pi \sin(r)$

euclidian geometry : $C = 2\pi r$

hyperbolic geometry : $C = 2\pi \sinh(r)$ → too big, means take a million times more to walk around a circle than it would to just across it.

, $2\sinh(r) \sim e^x$

AREA of a circle

spherical geometry $A = 2\pi(1-\cos(r))$

euclidian geometry $A = \pi r^2$

hyperbolic geometry $A = 2\pi(\cosh(r) - 1)$

Pythagorean theorem

spherical geometry $\cos a \cdot \cos b = \cos c$

euclidian geometry $a^2 + b^2 = c^2$

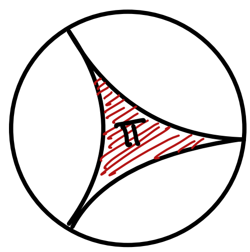
hyperbolic geometry $\cosh(a) \cdot \cosh(b) = \cosh(c)$

AREA of a Triangle

spherical geometry $\text{Area} = (A+B+C) - \pi$: sum of angles $- \pi$

euclidian geometry —

hyperbolic geometry $\text{Area} = \pi - (A+B+C)$: $\pi -$ sum of angles



$\pi - (0+0+0) = \pi$

Positive Definite Matrices

let A be a symmetric matrix in $M_n(\mathbb{R})$. Then A is pos-def if all it's eigenvalues are positive.

$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ $\operatorname{Tr}(A) = 1+5 = 6 = \lambda_1 + \lambda_2$, $\det(A) = \lambda_1 \lambda_2 = 1$ $\Rightarrow \lambda_1, \lambda_2 \Rightarrow$ posdef

if X and Y are two vectors in $\mathbb{R}^n$, we denote $\langle X, Y \rangle = X^T Y$

for any matrix A in $M_n(\mathbb{R})$: $\langle AX, Y \rangle = \langle X, A^T Y \rangle$

Thm Characterization of Pos-Def Matrices

Let A be symmetric matrix in $M_n(\mathbb{R})$ then A is pos-def iff

$$\langle x, Ax\rangle = x^T A x > 0 \quad \forall\, x \neq 0 \in \mathbb{R}^n$$

Since A is symmetric we have $\langle Ax, y\rangle = \langle x, A^T y\rangle = \langle x, Ay\rangle$

Pf assume A is pos-def, then the eigenvectors $v_1, v_2 \dots v_n$ of A form an orthonormal basis for $\mathbb{R}^n$. Thus λ_i is an eigenvalue of A and v_i is an assoc. eigenvector, then we have

$$\langle v_i, Av_i\rangle = \langle v_i, \lambda_i v_i\rangle = \lambda_i \langle v_i, v_i\rangle = \lambda_i \|v_i\|^2 = \lambda_i > 0, \quad i = 1, 2, \dots n$$

assume let λ be eigenvalue of A, ie $Ax = \lambda x$ for some eigenvector x assoc to λ.

now $\langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2 > 0 \Rightarrow \lambda > 0 \Rightarrow A$ is pos def

ex Show that the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is pos def

Thm says A is pos def iff $\langle x, Ax\rangle = x^T A x = \langle x, \lambda x\rangle = \lambda \|x\|^2 > 0$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\langle x, Ax\rangle = x^T A x$

$$= \underbrace{(x_1\ x_2\ x_3)}_{1\times 3} \underbrace{\big[\,A\,\big]}_{3\times 3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2x_1 - x_2 \quad -x_1 + 2x_2 - x_3 \quad -x_2 + 2x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2$$

$$= 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

$$\left(x_1 - \frac{x_2}{2}\right)^2 + \left(x_2 - \frac{x_3}{2}\right)^2 + \frac{3x_3^2}{4} - \frac{x_2^2}{4}$$

bu pozitif olamadı böyle gösteremedim.

OR we know $(x_1 - x_2)^2 \geq 0$

$$= x_1^2 - 2x_1x_2 + x_2^2 \geq 0$$

$$= -2x_1x_2 \geq -(x_1^2 + x_2^2)$$

$$= -x_1x_2 \geq \frac{-1}{2}(x_1^2 + x_2^2)$$

$$-x_1x_2 + x_1^2 + x_2^2 \geq \frac{-1}{2}(x_1^2 + x_2^2) + (x_1^2 + x_2^2) \geq 0$$

same for x_2x_3

$$-x_2x_3 \geq \frac{-1}{2}(x_2^2 + x_3^2)$$

$$\Downarrow$$

so $\langle x, Ax\rangle \geq 0 \Rightarrow A$ pos def

Thm if A pos def then A is invertible and A^{-1} is also pos def

Pf if A is pos def then all of its eigenvalues $\lambda_1, \dots, \lambda_n$ are positive

then we have

$$\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0$$ and this shows that A is invertible.

Now A^{-1} is symmetric since $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. Also, if λ is a positive eigenvalue of A then $\frac{1}{\lambda}$ is a positive eigenvalue of $A^{-1} \Rightarrow A^{-1}$ pos def

Let's go back to Hyperbolic Geometry:

a simple set of axioms for plane geometry can be represented in the framework of metric spaces. By a "line" in a metric space X we understand the image of a distance preserving map $\gamma: \mathbb{R} \to X$.

The 3 axioms of plane geometry are

incidence axiom (two distinct point, there pass a unique line"

reflection axiom (the complement of a given line in X has two connected components. There exists an isometry σ of X which fixes the points of the line but interchanges the two connected components of its complements.

parallel axiom (through a given point outside a given line there passes a unique line which does not intersect the given line.

Bolyai, Gauss, Lobachevsky found hyperbolic plane (H^2) which has the first two axioms but not parallel axiom. $\Rightarrow H^2$ is unique.

Classification thm a metric space which satisfies the 3 axioms of plane geometry is isometric to the Euclidian Plane. A space which satisfy the first 2 but not parallel axiom is isometric to H^2

Quadratic Forms

let k denote a field ($\mathbb{R}$, $\mathbb{C}$ and sometimes $\mathbb{Q}$) of characteristic $\neq 2$ and E a vector space over k of finite dimension n. By "quadratic form" on E we understand a fn $Q : E \to k$ homogenous of degree 2 ie

?

$$Q(\lambda z) = \lambda^2 Q(z) \qquad \lambda \in k \quad z \in E$$

with the property that the symbol

$$\underline{\langle x, y \rangle = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y) \right)} \qquad x, y \in E$$

polarization

is bilinear in x and y.

see that bilinear form $\langle x, y \rangle$ is "symmetrical" in x and y

?

17.07 Gar