Quadratic Forms

dot product in $\mathbb{R}^n$ our base field is $\mathbb{R}$ (sometimes $\mathbb{C}$)

$\vec{x}, \vec{y} \in \mathbb{R}^n$ $\vec{x} = (x_1, \dots, x_n)$ $\vec{x}\cdot\vec{y} = x_1y_1 + \dots + x_ny_n$

dot product:

$\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}$ $(\vec{x},\vec{y}) \mapsto \vec{x}\cdot\vec{y} = \langle \vec{x},\vec{y}\rangle$ bilinear map :

$\langle a\vec{x}+b\vec{y}, \vec{z}\rangle = a\langle\vec{x},\vec{z}\rangle + b\langle\vec{y},\vec{z}\rangle$

$\Rightarrow \langle\vec{x},\vec{y}\rangle = \langle\vec{y},\vec{x}\rangle$ (symmetric)

$\langle\vec{x},\vec{x}\rangle \geq 0$ with equality iff $\vec{x} = 0$ (positive definite, can be relaxed)

Define $Q(\vec{x}) = \langle\vec{x},\vec{x}\rangle$

quadratic form

$Q: \mathbb{R}^n \to \mathbb{R}$ fn st $Q(\lambda\vec{x}) = \lambda^2 Q(\vec{x}) = \langle\lambda\vec{x},\lambda\vec{x}\rangle = \lambda^2\langle\vec{x},\vec{x}\rangle$

$\langle\vec{x},\vec{y}\rangle = \frac{1}{2}\left[Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})\right]$ simetri nerde ?

: $(x+y)\cdot(x+y) - x^2 - y^2 \Rightarrow 2xy \cdot \frac{1}{2} = xy$: $\langle\vec{x}+\vec{y},\vec{x}+\vec{y}\rangle = \langle\vec{x},\vec{x}\rangle + 2\langle\vec{x},\vec{y}\rangle + \langle\vec{y},\vec{y}\rangle$

$Q(x) = x_1^2 + x_2^2 + \dots + x_n^2$

Defn a quadratic form on $\mathbb{R}^n$ is a fn $Q: \mathbb{R}^n \to \mathbb{R}$ st

$Q(\lambda\vec{x}) = \lambda^2 Q(\vec{x})$ for $\lambda \in \mathbb{R}$ $\vec{x} \in \mathbb{R}^n$

Define $\langle\vec{x},\vec{y}\rangle = \frac{1}{2}\left[Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})\right]$ polarization formula (pos def x)

↳ symmetric $\langle\vec{y},\vec{x}\rangle$ $\Updownarrow$ symmetric bilinear form (1-1 + onto)

Show bilinear $\langle a\vec{x}+b\vec{y}, \vec{z}\rangle = \frac{1}{2}\left[Q(a\vec{x}+b\vec{y}+\vec{z}) - Q(a\vec{x}+b\vec{y}) - Q(\vec{z})\right]$

$a\langle\vec{x},\vec{z}\rangle + b\langle\vec{y},\vec{z}\rangle = \frac{a}{2}\left[Q(\vec{x}+\vec{z}) - Q(\vec{x}) - Q(\vec{z})\right]$ … ?? (gereksiz)

Q'yu tanımlamak için unit sphereda

$Q(\vec{x})$: norm of $\vec{x}$ ~~standart norm $|\vec{x}| = \sqrt{x \cdot x}$~~

$\parallel$

$\langle \vec{x}, \vec{x} \rangle$

$\vec{x} \perp \vec{y}$ if $\langle \vec{x}, \vec{y} \rangle = 0$

Defn $U, V \subset \mathbb{R}^n$ subspaces

$U \perp V$ if $\langle \vec{u}, \vec{v} \rangle = 0$ for $\vec{u} \in U$ $\vec{v} \in V$ orthogonality

$K \subset \mathbb{R}^n$ vector space

$K^{\perp} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \perp \vec{k} \quad \forall \vec{k} \in K \}$ (total orthogonal subspace)

↳ is a vector subspace of $\mathbb{R}^n$

$\mathbb{R}^n = K \oplus K^{\perp}$

Q is non-singular if $(\mathbb{R}^n)^{\perp} = 0$

if $\langle \vec{x}, \vec{y} \rangle = 0 \quad \forall \vec{y} \in \mathbb{R}^n$ (tüm vektörlere dik tek vektör ?)

Q is singular = not non singular

ex we'll use $\mathbb{R}^4$, $\vec{x} = (x_1, x_2, x_3, x_4)$ → non singular

$Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$ → pos def $Q(\vec{x}) \geq 0$

↳ bilinear form $= 0$ iff $\vec{x} = 0$

suppose $\langle \vec{x}, \vec{y} \rangle = 0 \quad \forall \vec{y} \in \mathbb{R}^4$ non singular

$$\left.\begin{aligned} &\langle \vec{x}, \vec{e}_1 \rangle = x_1 = 0 \\ &e_1 = (1,0,0,0) \qquad e_2 = (0,1,0,0) \quad \cdots \\ &\langle \vec{x}_1, \vec{e}_2 \rangle = x_2 = 0 \end{aligned}\right\} \vec{x} = 0$$

ex $(\mathbb{R}^4)$ $\vec{x} = (x_1, x_2, x_3, x_4)$

$Q(x) = x_1^2 + x_2^2 + x_3^2 \qquad Q(\lambda x) = \lambda^2 Q(x)$

$$\langle x, y \rangle = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$$

$$= \frac{1}{2}\left((x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 - (x_1^2 + x_2^2 + x_3^2) - (y_1^2 + y_2^2 + y_3^2)\right) = x_1y_1 + x_2y_2 + x_3y_3$$

check if bilinear

$\langle ax+by, z\rangle = (ax_1+by_1)z_1 + (ax_2+by_2)z_2 + (ax_3+by_3)z_3$

$= a(x_1z_1 + x_2z_2 + x_3z_3) + b(y_1z_1 + y_2z_2 + y_3z_3)$

$= a\langle x,z\rangle + b\langle y,z\rangle$ bilinear ✓ $\Rightarrow$ so Θ is a quadratic form on $\mathbb{R}^4$

$\Theta((0,0,0,1)) = 0$

but $(0,0,0,1) \neq 0$

for all $\vec{x} \in \mathbb{R}^4$, $\Theta(x) \geq 0$

$\langle (0,0,0,1), (x_1, x_2, x_3, x_4)\rangle = 0 \quad \forall \vec{x} \in \mathbb{R}^4$, so $(0,0,0,1) \in (\mathbb{R}^4)^{\perp}$

so Θ is not non singular $\Rightarrow$ Θ is singular

ex $(\mathbb{R}^4)$ $\Theta(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$

$\langle x,y\rangle = \frac{1}{2}[\Theta(x+y) - \Theta(x) - \Theta(y)] =$

$$= \frac{1}{2}\left[(x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 - (x_4+y_4)^2 - (x_1^2+x_2^2+x_3^2-x_4^2) - (y_1^2+y_2^2+y_3^2-y_4^2)\right.$$

$= x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ ✓ symmetric, bilinear

$\Theta((0,0,0,1)) = -1$ so Θ is not pos def.

check singularity

suppose $\langle \vec{x}, \vec{y}\rangle = 0 \quad \forall y \in \mathbb{R}^4$

$$\left.\begin{aligned} \langle \vec{x_1}\ \vec{e_1}\rangle &= x_1 = 0 \\ \langle \vec{x_2}\ \vec{e_2}\rangle &= x_2 = 0 \\ \langle \vec{x_3}, \vec{e_3}\rangle &= x_3 = 0 \end{aligned}\right\} \rightarrow \vec{x} = 0$$

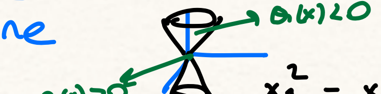
$\langle \vec{x_4}, \vec{e_4} = -x_4 = 0 = x_4 = 0$

so Θ is non singular □

for which vectors $\vec{x}$ is $\Theta(\vec{x}) = 0$ → isotropic (norm = 0 or $\Theta(x) = 0$)

$\underbrace{x_1^2 + x_2^2 + x_3^2 - x_4^2}_{\text{cone}} = 0$ OR $x_4^2 = x_1^2 + x_2^2 + x_3^2$

$x_1^2 + x_2^2 - x_3^2 \rightarrow x_3^2 - x_1^2 - x_2^2$ yazılabilir



$x_3^2 = x_1^2 + x_2^2$

konige C de $\mathbb{R}^3 \setminus C$ has 3 components ? işaret değiştir if $Q(x) = 1$ $x_1^2 + x_2^2 - x_3^2 = 1$ tek yaprak hiperbloid 

$Q(x) = -1$ $x_1^2 + x_2^2 - x_3^2 = -1$ cift yaprak // hiperbolik düzlem insası

Thm Q non singular on $\mathbb{R}^n$ $K \subset \mathbb{R}^n$ subspace

Then $\dim K + \dim K^\perp = n$ proof exercise Iversen

Corollary $\mathbb{R}^n = K \oplus K^\perp$ suppose $0 \neq \vec{x} \in K$ is not isotropic

then $\mathbb{R}^n = K \oplus K^\perp$

$\vec{x} \in K \cap K^\perp$ $\langle x, x \rangle = Q(x) = 0 \Rightarrow \vec{x} = 0$

isomorphism (1-1 + onto)

$(\mathbb{R}^n, Q) \overset{\cong}{\to} (\mathbb{R}^n, Q')$ is a vector space isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ st

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "\varphi"] \arrow[dr, "Q'"'] & & \mathbb{R}^n \arrow[dl, "Q"] \\
& \mathbb{R} &
\end{tikzcd}$$

$Q \circ \varphi = Q'$

commutes.

bu isomorphismin tersi de var. → φ^{-1} $Q' \circ (\varphi^{-1}) = Q$

if there exist isomorphism then there is also automorphism

↳ objeden kendisine isomorphism

automorphism isomorphism of an object to itself

$(\mathbb{R}^n, Q) \xrightarrow{\varphi} (\mathbb{R}^n, Q)$

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$ st $Q \circ \varphi = Q$

automorphism of any object form a group under composition

$X = \{1, 2, 3\}$

$Aut(X) = S_3$ $123 \to 123$ herbiri bir kez $Aut(X) \neq 0$ $id \in Aut(X)$

$\varphi, \psi \in Aut(X)$ $X \xrightarrow{\varphi} X \xrightarrow{\psi} X$ $\varphi \in Aut(X) \Rightarrow \varphi^{-1} \in Aut(X)$ *

$(Q \circ \varphi) \circ \varphi^{-1} = Q \circ \varphi^{-1} \Rightarrow Q \circ \varphi^{-1} = Q$

$(\mathbb{R}^n, Q)$

$\mathrm{Aut}((\mathbb{R}^n, Q)) = O(Q)$ transpose corp identity

$\mathbb{R}^2 \qquad Q(x) = x_1^2 + x_2^2$

$\langle x, y\rangle = x_1y_1 + x_2y_2 \quad , \quad (\mathbb{R}^2, Q) \xrightarrow[\cong]{\varphi} (\mathbb{R}^2, Q) \qquad \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \qquad Q \circ \varphi = Q$

$\varphi : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; ad - bc \neq 0 \qquad \varphi(x_1, x_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$

$= (ax_1 + bx_2 \, , \, cx_1 + dx_2)$

ama φ Q'yu korumak $\varphi \circ Q = Q$

$$(ax_1 + bx_2)^2 + (cx_1 + dx_2)^2 = x_1^2 + x_2^2$$

$$a^2x_1^2 + b^2x_2^2 + 2abx_1x_2 + c^2x_1^2 + d^2x_2^2 + 2cx_1dx_2 = x_1^2 + x_2^2$$

$$(a^2 + c^2)x_1^2 + (b^2 + d^2)x_2^2 + (2ab + 2cd)x_1x_2 = x_1^2 + x_2^2$$

$$\begin{matrix} a^2 + c^2 = 1 \\ b^2 + d^2 = 1 \\ ab + cd = 0 \end{matrix} \iff \underset{I_{2\times 2}}{\overset{}{A . A^T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}$$

$O((\mathbb{R}^2, Q)) = O(2, \mathbb{R})$

2x2 (2 dimensional)

Gram Matrix

$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad Q : \mathbb{R}^n \to \mathbb{R}$

pick an ordered basis for $\mathbb{R}^n$

$\vec{e}_1, \vec{e}_2 \, ..., \vec{e}_n$

$\langle e_i, e_j\rangle = G_{ij} \quad , \quad G = [G_{ij}]_{i,j}^{n}$ gram matrix

$\vec{x} = x_1\vec{e}_1 + ... + x_n\vec{e}_n = \sum x_ie_i$

$y = y_1\vec{e}_1 + ... + y_n\vec{e}_n = \sum y_je_j$

gram matrix symmetric

$G_{ij} = G_{ji}$

$$\langle \vec{x}, \vec{y}\rangle = \langle \sum_{i=1}^{n} x_i\vec{e}_i \, , \sum_{j=1}^{n} y_j\vec{e}_j \rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} x_iy_j \underbrace{\langle e_i, e_j\rangle}$$

ex $(\mathbb{R}^2)$ $n=2$ $\langle , \rangle$ Θ e_1, e_2

$G_{ij} = \langle e_i, e_j \rangle$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} , \quad \left.\begin{array}{l} x = x_1 e_1 + x_2 e_2 \\ y = y_1 e_1 + y_2 e_2 \end{array}\right\} \rightarrow \begin{array}{l} x : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y : \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{array}$$

$G_{12} = G_{21}$

$$x^T . G . y = [x_1 \ x_2] \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =$$

$$[x_1 \ x_2] \begin{bmatrix} G_{11} y_1 + G_{12} y_2 \\ G_{21} y_1 + G_{22} y_2 \end{bmatrix} =$$

$$= x_1 G_{11} y_1 + x_1 G_{12} y_2 + x_2 G_{21} y_1 + x_2 G_{22} y_2 \qquad 1 \times 1$$

Orthonormal Basis an orthonormal basis for $\mathbb{R}^n$ is a basis $\vec{e_1}, \dots, \vec{e_n}$ st

$\langle e_i, e_j \rangle = 0$ if $i \neq j$

$\langle e_i, e_j \rangle \in \{\mp 1, 0\}$ if $i = j$ (-1 olur cünkü $x_1^2 - x_2^2 - x_3^2$ de quadratic form)

there exist an orthonormal basis for $(\mathbb{R}^n, \Theta)$

pf if $\Theta = 0$, any basis is orthonormal $\Theta = 0$ da quadratic form

if $\Theta \neq 0$, pick $\vec{e_1} \in \mathbb{R}^n$ st $\Theta(\vec{e_1}) = \langle e_1, e_1 \rangle = \mp 1$

use induction to find an orthonormal basis for $(\mathbb{R}, e_1)^\perp$

$$\mathbb{R}^n = (\mathbb{R} e_1) \oplus (\mathbb{R} e_1)^\perp$$

$$\vec{e_1} \qquad \vec{e_2}, \dots \vec{e_n} \longrightarrow \vec{e_1}, \vec{e_2} \dots \vec{e_n}$$?

$$\begin{bmatrix} \ddots & & 0 \\ & \mp 1, 0 & \\ 0 & & \ddots \end{bmatrix}$$ diagonlize yapıyoruz.

the number of 1's 0's -1's are independent of the orthonormal basis

see proof in iversen

Q is non singular iff # 0's on diagonal is 0 *

$\mathbb{R}^4$, $Q = x_1^2 + x_2^2 + x_3^2 \rightarrow$ singular

$(-0,3)$ → -1'in sayısı 0 tane (-1 yok)

?

→ +1'lein sayısı 3 tane

$\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}$ standart basis

$$G = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

$\mathbb{R}^4$, $Q = x_1^2 + x_2^2 + x_3^2 - x_4^2$ $\quad G = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ (read prop 1.11 iversen)

$(-1,3)$?

Prop 1.11 Q non singular $\sigma \in O(Q)$ has $\det \in \{\mp 1\}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "\sigma"] \arrow[dr, "Q"'] & & \mathbb{R}^n \arrow[dl, "Q"] \\ & \mathbb{R} & \end{tikzcd}$$

$\sigma : \mathbb{R}^n \xrightarrow{\sigma} \mathbb{R}^n$

$SO(Q) = \{\sigma \in O(Q) \mid \det \sigma = 1\}$
↳ special orthogonal group

$[O(Q) : SO(Q)] = 2$ (index)

Sylvester Types

if all 1's no zero pos def
if all -1's no zero neg def

$(\mathbb{R}^n, Q)$

$$G = \begin{bmatrix} \underbrace{1 \cdots 1}_{q} & \underbrace{-1 \cdots -1}_{p} & \underbrace{0 \cdots 0}_{r} \end{bmatrix}$$

has sylvester type $(-p, q)$

$p + q + r = n$

if # 0's is 0 (if Q is non singular) $\quad \det G = (-1)^p$

Q is pos def if $Q(\vec{x}) > 0$ for $\vec{x} \neq 0$ ($\mathbb{R}^n$ euclidean space)

Q is neg def if $Q(\vec{x}) < 0$ for $\vec{x} \neq 0$

pos/neg def $\Rightarrow$ non singular

Self note observe that two quadratic forms (E, Q) and (F, R) with $\dim E = \dim F$ of equal Sylvester types are isomorphic.

Dimension Formula for subspaces U and V of the finite dimensional vector space E we have : $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

Pf apply Grassman dim formula $f : U \oplus V \rightarrow E$, $f(u,v) = u - v$ observe $\operatorname{Im} f = U + V$, $\operatorname{Ker} f \rightarrow U \cap V$

Lecture Note (22.07)

Recall $Q : \mathbb{R}^n \to \mathbb{R}$

Polarization

$Q(2x) - Q(x)$

$Q(\lambda \vec{x}) = \lambda^2 Q(\vec{x})$ s.t. $\langle x, y \rangle = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$ is bilinear

$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ symmetric bilinear , $Q(x) = \langle x, x \rangle$

ex $Q : \mathbb{R}^3 \to \mathbb{R}$ quad form

$$Q(x,y,z) = x^2 + 2y^2 + 5z^2 + 2xy - 4xz + 8yz$$

$G = \langle e_i, e_j \rangle$ $[G_{ij}] = G$

let e_1, e_2, e_3 be standart basis of $\mathbb{R}^3$

$$\langle x, y \rangle = x^T G y = \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij} x_i y_j$$

Gram matrix :

$x^2 + x + 1 \to$ homojen değil

$x^2 + xy + y^2 \to$ homojen

$\langle e_1, e_1 \rangle = Q(e_1) = Q(1,0,0) = 1$

$\langle e_2, e_2 \rangle = Q(e_2) = Q(0,1,0) = 2$

$\langle e_3, e_3 \rangle = Q(e_3) = Q(0,0,1) = 5$

$\langle e_1, e_2 \rangle = \frac{1}{2}[Q(1,1,0) - Q(1,0,0) - Q(0,1,0)] = \frac{1}{2}(5-1-2) = 1$

$\langle e_1, e_3 \rangle = \frac{1}{2}[Q(1,0,1) - Q(1,0,0) - Q(0,0,1)] = \frac{1}{2}(2-1-5) = -2$

$\langle e_2, e_3 \rangle = \frac{1}{2}[Q(0,1,1) - Q(0,1,0) - Q(0,0,1)] = \frac{1}{2}(15-2-5) = 4$

$$G = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$ diagonal'lar köşede xy/yz/xz katsayılarının yarısı simetrik dağıt.

$\Downarrow$

$$\det(G) = \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = -35$$

$\det(G) \neq 0 \Rightarrow Q$ is non singular

$\langle x, y \rangle = 0 \quad \forall y \in \mathbb{R}^3 \Rightarrow \vec{x} = 0$

check $[x\ y\ z]\ G \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q(x,y,z)$

Sylvester Criteria matrise bakarak pos def / neg def anlama

remember $Q : \mathbb{R}^n \to \mathbb{R}$ quad form pos-def if $Q(\vec{x}) > 0 \quad \forall x \neq 0$
neg-def if $Q(x) < 0 \quad \forall x \neq 0$

pos/neg def $\underset{\not\Leftarrow}{\Rightarrow}$ non singular

M : $n \times n$ symmetric over $\mathbb{R}$

$w_i = \det M_i$ $\quad M_i$: top left $i \times i$ block of M $\quad i = 1, \dots, n$ $\quad$ sol köşeden 1×1 2×2 . . . $n \times n$ bak

M pos-def iff all $w_i > 0$, M neg def iff $w_1 \; w_3 \; w_5 < 0$
$w_2 \; w_4 \; w_6 > 0$

note Q pos def iff $-Q$ neg def

$w_{2,4,6}$ aynen kalıyor $w_{1\,3\,5}$ işaret değişiyor neg def bakarken

matrisi (−) ile çarpınca işaret değişiyor.

$w_1 = 1 > 0$ $\qquad w_3 = -35 < 0$

$w_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0$ $\qquad$ so Q is neither pos def nor neg def

$$\cancel{\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}} \qquad \overset{\checkmark}{\begin{matrix} 1 & & \\ & 1 & \\ & & -1 \end{matrix}} \qquad \underset{\substack{\downarrow \\ \text{not this bec of det}}}{\cancel{\begin{matrix} 1 & & \\ & -1 & \\ & & -1 \end{matrix}}} \qquad \cancel{\begin{matrix} -1 & & \\ & -1 & \\ & & -1 \end{matrix}}$$

Q has Sylvester Type $(-1, 2)$ $\quad$ eleme olmadan orthonormal basis bul açıkla

norm $\neq 0$ pick a vector norm $= \mp 1$ $\quad$ boyut $3 \to 2$ $\quad$ use induction

1) $Q(e_1) = 1 \qquad f_1 = e_1$

2) $(\mathbb{R} f_1)^{\perp} = \{ x \in \mathbb{R}^3 \mid \langle x, f_1 \rangle = 0 \}$

$x = a_1 \vec{e_1} + a_2 \vec{e_2} + a_3 \vec{e_3}$, $\quad \langle x, f_1 \rangle = 0 : a_1 \overset{1}{\langle e_1, e_1 \rangle} + a_2 \overset{1}{\langle e_2, e_1 \rangle} + a_3 \overset{-2}{\langle e_3, e_1 \rangle}$
$= a_1 + a_2 - 2a_3 = 0$

$\Rightarrow (\mathbb{R} f_1)^{\perp} = \{ x = a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{R}^3 \mid a_1 + a_2 - 2a_3 \} = 0$

$Q(1,1,1) = 14$ $\quad$ choose $f_2 = \frac{1}{\sqrt{14}} (e_1 + e_2 + e_3)$ then $Q(f_2) = 1 \quad \in (\mathbb{R} f_1)^{\perp}$

$(\mathbb{R} f_1 \oplus \mathbb{R} f_2)^{\perp} = \{ x = a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{R}^3 \mid a_1 + a_2 - 2a_3 = 0 , \; a_2 + a_3 = 0 \}$

$Q(3,-1,1) = -10$ $\quad$ choose $f_3 = \frac{1}{\sqrt{10}} (3e_1 - e_2 + e_3) \Rightarrow Q(f_3) = -1 \;\Big|\; f_1 f_2 f_3$ gram $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Discriminant Inequality let (E, Q) be a non singular quadratic form of Sylvester type $(-p, q)$. For any basis $e_1, \dots, e_n$ for E we have

$$\underline{\text{sign}\ \det}_{ij} \langle e_i, e_j \rangle = (-1)^p$$

bu ne demek ?

Question: if trace is zero then any 2x2 matrices have

$$\det A = \langle A, A \rangle$$

benzerlik niye?

$$\underbrace{\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix}}_{\langle A, A\rangle\ ?} = (a^2+bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq -a^2 - bc$$

Euclidean Vector Spaces

$(\mathbb{R}^n, Q)$ pos def, a vector $\vec{x}$ in E has length

$|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$, gram matrix of Q wrt any orthonormal basis $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \ddots 1 \end{bmatrix}$

we know $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots$

so $\langle x, x \rangle = x_1^2 + x_2^2 + \dots + x_n^2$

euclidean vector space is finite dimensional real vector space equipped with pos def quad.

Cauchy Schwarz $|\langle x, y \rangle| \leq |\vec{x}| \cdot |\vec{y}|$ → this is $<$ if x, y linearly independent

pf trivial if x and y are linearly dependent.

if linearly independent, they span Euclidean plane and we get from discriminant inequality, sylvester type $(-p, q)$, for any basis $e_1, \dots e_n$

we had $\text{sign} \det_{ij} \langle e_i, e_j \rangle = (-1)^p$ so with this we get

$$\det \begin{bmatrix} \overset{x^2}{\langle x, x \rangle} & \overset{xy}{\langle x, y \rangle} \\ \underset{xy}{\langle y, x \rangle} & \underset{y^2}{\langle y, y \rangle} \end{bmatrix} > 0$$

$x^2 + y^2 - 2xy \to (x-y)^2 > 0$?

Triangle inequality $|\vec{x}+\vec{y}| \leq |\vec{x}|+|\vec{y}|$ [also $\cos\theta = \dfrac{\langle \vec{x},\vec{y}\rangle}{|\vec{x}|\cdot|\vec{y}|}$]

↳ this is < if x and y are linearly independent.

pf look at $|\vec{x}+\vec{y}|^2 = |x|^2 + |y|^2 + \underbrace{2|x||y| + 2(\langle x,y\rangle - |x||y|)}_{\leq 2|x||y|}$

we know $\langle x,y\rangle < |x||y|$

$= \langle x,y\rangle - |x||y| < 0$

$= 2\langle x,y\rangle - 2|x||y| < 0$

$= 2\langle x,y\rangle - 2|x||y| + 2|x||y| < 2|x||y|$

$\leq 2|x||y|$

what is $(|\vec{x}| + |\vec{y}|)^2$?

we said that the $\cos\theta = \dfrac{\langle \vec{x},\vec{y}\rangle}{|\vec{x}|\cdot|\vec{y}|}$ θ can be written as $\angle(x,y)$

Reflection in a linear hyperplane H with unit normal vector $\vec{n} \in \mathbb{R}^n$:

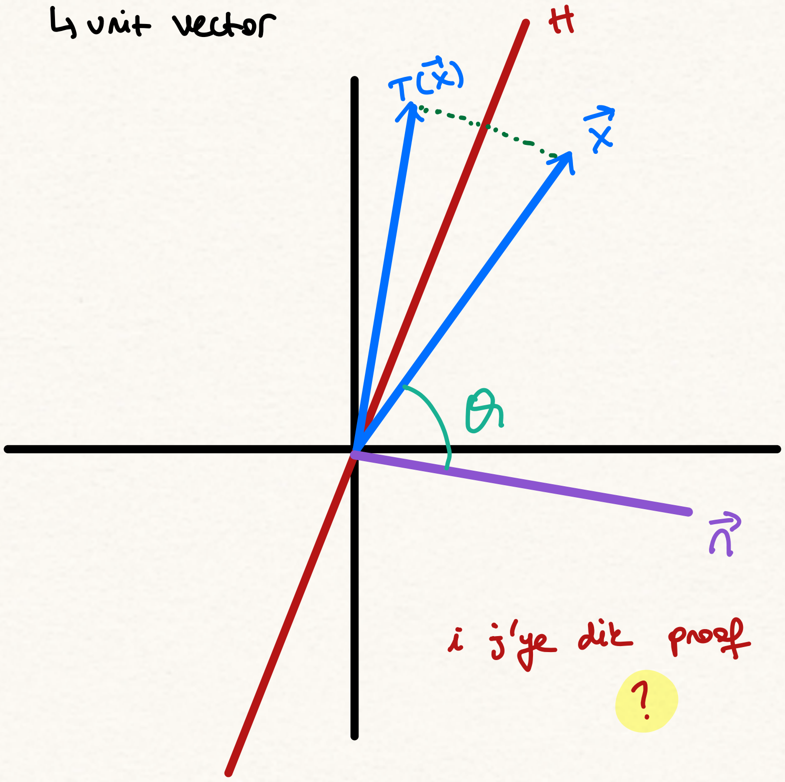
$T(\vec{x}) = \vec{x} - 2\langle \vec{x},\vec{n}\rangle\cdot\vec{n}$ $\quad T \in O(\vec{n})$, $\det(T) = -1$

and $\langle \vec{x},\vec{n}\rangle = |\vec{x}|\cdot\cos\theta$ is the projection of $\vec{x}$ along $\vec{n}$

↳ $\angle(x,n)$

$|n| = 1$

↳ unit vector



i j'ye dik proof nasil ?

?

⇒ $\sigma \in O(n)$ is the product of at most n reflections

⇒ $O(2)$ in $\mathbb{R}^2$. Pick an orthonormal basis $\hat{i}, \hat{j}$ and let $\sigma \in O(2)$ act as:

$\sigma(\hat{i}) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\sigma(\hat{j})$ is one of the unit vectors perpendicular ($\perp$) to $\sigma(i)$.

$\sigma(j) = \mp(-\sin\theta\,\hat{i} + \cos\theta\, j)$

one gives rotation other reflection

$T_n : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation $\quad T(\vec{x}) = \vec{x} - 2\langle \vec{x}, \vec{n} \rangle \cdot \vec{n}$

1) $T_n(x+y) = x+y - 2\langle x+y, n\rangle n$

$$= x+y-2\left(\langle x,n\rangle + \langle y,n\rangle\right) n$$

$$= \left[x-2\langle x,n\rangle n\right] + \left[y - 2\langle y,n\rangle n\right]$$

$$= T_n(x) + T_n(y)$$

2) $T(\lambda x) = \lambda x - 2\langle \lambda x, n\rangle n$

$$= \lambda x - 2\lambda \langle x,n\rangle n$$

$$= \lambda\left[x - 2\langle x,n\rangle n\right]$$

$$= \lambda T_n(x)$$

what happens if $T_n(T_n(x))$?

$$T_n(T_n(x)) = T_n(x - 2\langle x,n\rangle n)$$

$$= x - 2\langle x,n\rangle n - 2\langle x - 2\langle x,n\rangle, n\rangle n$$

$$= x - 2\langle x,n\rangle n - 2\left(\langle x,n\rangle - 2\langle x,n\rangle \overset{1}{\overset{\shortparallel}{\langle n,n\rangle}}\right) n$$

$$= x - 2\langle x,n\rangle n + 2\langle x,n\rangle = x \qquad \text{so} \quad T_n(T_n(x)) = x$$

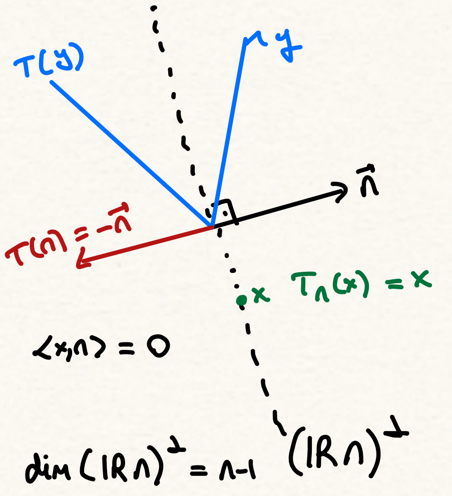
T_n involution 2 kere al kendisine dönün

what about $T_n(n)$?

$$T_n(n) = n - 2\langle n,n\rangle n$$

$$= n - 2n = -n \qquad T_n \neq id$$

T_n (geometrically) : T_n is the reflection in the hyperplane passing through the origin and having unit normal n.



Pick a basis $\{f_2, \dots f_n\}$ for $(\mathbb{R}n)^\perp$

$f_1 = n$, then $\{f_1, f_2, \dots f_n\}$ is a basis for $\mathbb{R}^n$

$$T_n = \begin{bmatrix} -1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \\ & & & & & 1 \end{bmatrix} \Rightarrow \det(T_n) = -1$$

$O(\mathbb{R}^n, \Theta) = O(n)$: orthogonal group

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\overset{\sigma}{\varphi}"] \arrow[dr, "\Theta"'] & \mathbb{R}^n \arrow[d, "\Theta"] \\ & R \end{tikzcd}$$

$\Theta \circ \varphi = \Theta$

an element $\sigma \in O(n)$ is the product of at most n reflections

$$\det \sigma = \{1, -1\}$$

$\swarrow$ $+1$ $\searrow$ -1

σ: product of even # of refl only

σ: product of odd # of reflections only

$SO(n) = \{\sigma \in O(n) \mid \det \sigma = 1\}$ special orth.

subgroup of $O(n)$

$$[O(n) : SO(n)] = 2 \qquad SO(n) \trianglelefteq O(n)$$

$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ unit sphere

$O(n)$ and $SO(n)$ act on S^{n-1}

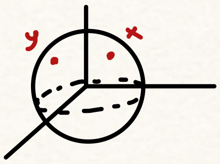
$\sigma \in O(n)$, $x \in S^{n-1}$

$Q(x) = 1 = Q(\sigma(x))$, so $\sigma(x) \in S^{n-1}$

$O(n) \times S^{n-1} \to S^{n-1}$

$(\sigma, x) \mapsto \sigma(x)$ $\qquad id(x) = x$ $\qquad O(n)$ acts on S^{n-1}

$(\sigma T)(x) = \sigma(T(x))$



there is $\sigma \in O(n)$ st $\sigma(\vec{x}) = \vec{y}$

Group Action : a group action G acts on a set $X \neq \emptyset$ (from the left)

if there is a map $G \times X \to X$

$(g, x) \to g \cdot x$ st $\overset{*}{ex = x}$ $\quad \overset{*}{(g.h)x = g(hx)}$

$G \to Sym(X)$ [group homomorphism]

$O(n)$, $SO(n)$ acts on S^{n-1} transitively $\forall x, y \in S^{n-1}$ $\exists \sigma \in O(n)$ st

$\in SO(n)$

$$\sigma(x) = \sigma \cdot \vec{x} = \vec{y}$$

Euclidean Plane $\mathbb{R}^2$ $\quad Q(x,y) = x^2+y^2$

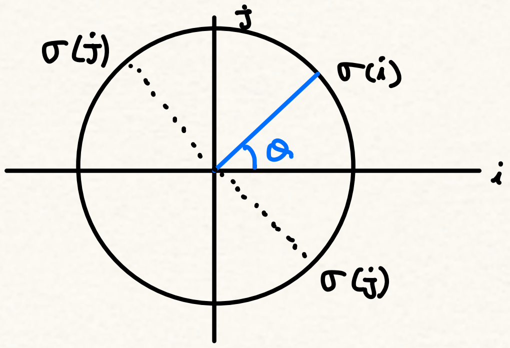
i, j unit vector : orthonormal basis

$\langle i,i\rangle = \langle j,j\rangle = 1$ but $\langle i,j\rangle = 0$ $\qquad i,j \in S^1 \qquad \sigma(i) \in S^1$

let $\sigma \in O(2)$

$\sigma(i) = \cos\theta\, i + \sin\theta\, j$

$\sigma(j) = \mp\left[(-\sin\theta)\, i + \cos\theta\, j\right]$



$\langle \sigma(i), \sigma(j)\rangle = \langle i,j\rangle = 0$

↓

unit vector perp to $\sigma(i)$

+ sign:

$$\underbrace{\sigma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by } \theta \text{ about origin}} : \det = 1$$

− sign:

$$\underbrace{\sigma = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{\text{reflection along } y = \tan\left(\frac{\theta}{2}\right)x} : \det = -1$$

$SO(2)$ rotation about origin, $\sigma_\theta \cdot \sigma_{\theta'} = \sigma_{(\theta+\theta')}$

$SO(2) \cong \mathbb{R}/2\pi\mathbb{Z}$ (quotient group)

$\qquad \cong S^1 \pmod{2\pi}$ [circle group]

anlarsın → $O(3)$ det -1 $\qquad$ anlarsın

$\mathbb{R}^3$ $\quad Q(x,y,z) = x^2+y^2+z^2$ $\qquad \sigma \in SO(3)$ is a rotation

ι (iota) (birim matrix) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$?

$\det(\sigma) = 1 \quad , \quad x(t) = \det(\iota t - \sigma)$

characteristic polynomial

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{vmatrix} t-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & t-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & t-a_{33} \end{vmatrix} = \overbrace{t^3 - (\operatorname{tr}\sigma)t^2 + kt - \det\sigma}^{\text{char polynomial}}$$

$$k = (a_{11}+a_{22}+a_{33})$$

$x(t) = \det(\iota t - \sigma)$ has a real root λ

$\lambda = \mp 1$

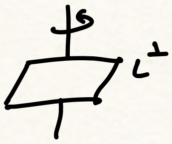
pick $0 \neq \vec{e}$ with $\sigma(e) = \lambda e$ $\qquad |e| = |\sigma(e)| = |\lambda|\,|e|$

if $\det(\sigma) = 1$ $\qquad x(t) = t^3 \dots -1$, $x(0) = -1$, $\lim\limits_{t\to\infty} x(t) = \infty$, $x(1) = 0$, $\sigma(e) = e$

take $\mathcal{L} = \mathbb{R}\vec{e}$, $\mathbb{R}^3 = \mathcal{L} \oplus \mathcal{L}^\perp$

dim $\mathcal{L} = 2$, pick an orthonormal basis f, g for $\mathcal{L}^\perp$

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



σ : one real , two complex conjugate eigenvalues

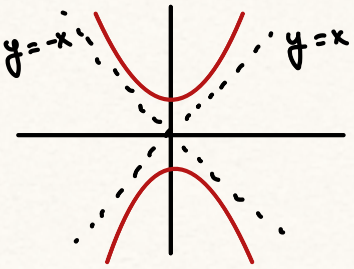
Lorentz Group $\mathbb{R}^{n+1}$, Q has Sylvester Type $(-n,1)$

$$Q(x_1,\dots,x_{n+1}) = -x_1^2 - \dots - x_n^2 + x_{n+1}^2 \implies G = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & -1 & \\ & & & +1 \end{bmatrix} \quad \det G = -1$$

$S = \{ x \in \mathbb{R}^{n+1} \mid \langle x,x \rangle = 1 \}$ (hyperboloid)

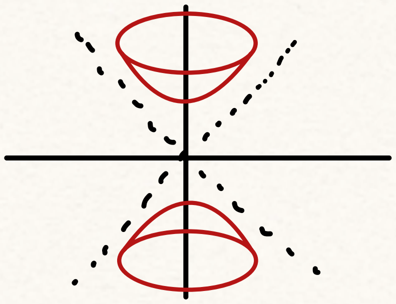
$n:0$ no line

$n:1$ $Q(x,y) = -x^2 + y^2 = 1$



hyperboloid $-x^2+y^2=1$

$n:2$ $Q(x,y,z) = -x^2-y^2+z^2$ $S: -x^2-y^2+z^2=1$



hyperboloid of two sheets

$n \geq 2$
two components

prop 2.3 Q be quad form on $\mathbb{R}^n$ for $\lambda \in \mathbb{R}^*$

$O(Q)$ acts transitevely on $\{ x \in \mathbb{R}^n \mid Q(x) = \lambda \}$

Pf pick $x,y \in \mathbb{R}^n$, $Q(x) = Q(y) = \lambda$

$$\langle x-y, x+y \rangle = \langle x,x \rangle + \cancel{\langle x,y \rangle} - \cancel{\langle y,x \rangle} - \langle y,y \rangle = \lambda - \lambda = 0$$

this means we can NOT have $Q(x+y)$ and $Q(x-y) = 0$ at the same time.

$$x = \frac{1}{2}(x+y) + \frac{1}{2}(x-y)$$

$$\langle x,x \rangle = \frac{1}{4}\langle x+y, x+y \rangle + \frac{1}{4}\langle x+y, x-y \rangle + \frac{1}{4}\langle x-y, x+y \rangle + \frac{1}{4}\langle x-y, x-y \rangle$$

$\Downarrow$

$\lambda \neq 0$

we found $Q(x+y)$ and $Q(x-y)$ can NOT be zero at the same time

case I: $Q(x-y) \neq 0$

T: reflection along $\widehat{x-y}$ $$T(x) = x - 2\frac{\langle x, n\rangle}{\langle n, n\rangle} n$$

$T(x-y) = y-x$ (reflection -1'e götürsyor)

$$T(x+y) = x+y-2\frac{\langle x+y, x-y\rangle}{\langle x-y, x-y\rangle}\cdot(x-y) = x+y$$

so $T(x-y) = y-x$, $T(x+y) = x+y \Rightarrow T(2x) = 2y \Rightarrow T(x) = y$

case II: $Q(x+y) \neq 0$

P: reflection along $x+y \in O(n)$ $$P(x) = x-2\frac{\langle x, x+y\rangle}{\langle x+y, x+y\rangle}(x+y)$$

$$P(x-y) = x-y-2\frac{\langle x-y, x+y\rangle}{\langle x+y, x+y\rangle}(x+y) = x-y$$

$P(x+y) = -x-y$

so $P(x-y+x+y) = -2y \Rightarrow P(x) = -y$ $\boxed{T_y \circ P(x) = y}$

back to Lorentz Group

$\mathbb{R}^{n+1}$ $Q(x_1 \ldots x_{n+1}) = -x_1^2 - \cdots - x_n^2 + x_{n+1}^2$

$S = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = 1\}$ hyperboloid of 2 sheets $O(Q)$ acts transitively on S

let $x, u \in S \Rightarrow |\langle x, u\rangle| \geq 1$ ($>$ if x, u lin. indep)

Discriminant Lemma let R be a plane $R \subset \mathbb{R}^{n+1}$ generated by x, y (lin indep)

$\Delta = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2$ (det of Gram Matrix)

$\Delta < 0$	$\Delta = 0$	$\Delta > 0$	
$(-1,1)$	$(-1,0)$	$(-2,0)$	Sylvester Type

let $E \subset \mathbb{R}^{n+1}$ be a subspace with Sylvester type $(-n, 0)$

pick $0 \neq x \in E \cap R$, $Q(x) < 0$ so
R always contain non zero vector with (-) norm.

$\dim E = n$, $E \subset \mathbb{R}^{n+1}$ $R \subset \mathbb{R}^{n+1}$ $\dim R = 2$

$$\underbrace{\dim E + \dim R}_{n+2} = \underbrace{\dim(E+R)}_{\leq n+1} - \underbrace{\dim(E\cap R)}_{\geq 1}$$

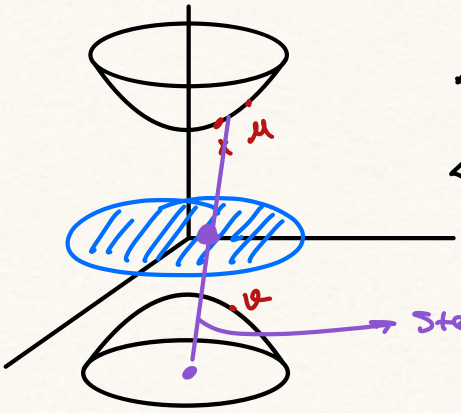
R: $\cancel{{}^{0}{}_{0}}$ $\cancel{{}^{1}{}_{0}}$ $\cancel{{}^{1}{}_{1}}$ ${}^{-1}{}_{0}$ ${}^{-1}{}_{-1}$ ${}^{-2}{}_{0}$???

$\Delta = 0$ $\Delta < 0$ $\Delta > 0$

Prop *** $S: -x_1^2 - \dots - x_n^2 + x_{n+1}^2 = 1$ has two connected components

let $x, u \in S$ are in the same connected component iff $\langle x, u \rangle > 0$

$n=2$ $\mathbb{R}^3$ $Q(x,y,z) = -x^2 - y^2 + z^2$ $S: -x^2 - y^2 + z^2 = 1$ $x^2 + y^2 = z^2$ asimptot



$\langle x, u \rangle > 0$

$\langle x, v \rangle < 0$

Stereographic Projection $D : (x,y,z)$ $z = 0$
$x^2 + y^2 < 1$

BOOK NOTES

Recall an orthagonal transformation has $\det = \mp 1$. It follows that the special orthagonal group

$$SO(n) = \{\sigma \in O(n) \mid \det \sigma = 1\}$$ has index 2 in $O(n)$

we have seen that $O(n)$ operates transitively on the sphere

$$S(n) = \{e \in n \mid |e| = 1\}$$

how does $SO(n)$ acts transitively on $S(n)$, $\dim(n) \geq 2$?

Euclidean Plane $\mathbb{R}^2$, $Q(x,y) = x^2+y^2$

Assume $\dim(n) = 2$

An orthagonal transformation of E with $\det = 1$ called rotation

we said $\sigma(i) = i\cos\theta + j\sin\theta$

and since i is orthagonal to j, these are the two chances of σ matrices.

$$\text{ROTATION} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \text{REFLECTION}$$

$$\Downarrow$$

$$\det : \underbrace{\cos^2\theta + \sin^2\theta}_{1} \qquad \det : \underbrace{-\cos^2\theta - \sin^2\theta}_{-1}$$

the $\sigma(i)$ and $\sigma(j)$ [addition formulas for trigo fns] that the matrix with $\det : 1$ defines an isomorphism between the circle group

Lemma let $\sigma : E \to E$ be an endomorphism ? of a finite dimensional real vector space E. $\exists$ subspace $R \subset E$ of $\dim = 1$ or 2 stable under σ

Pf [complexification of σ]

we know $E_{\mathbb{C}} = E \oplus E$ turn this into vector space over $\mathbb{C}$ through convention

$$(x+iy)\cdot(e,f) = (xe - yf, xf + ye) \qquad x,y \in \mathbb{R} \qquad e,f \in E$$

how did this happen

identify $e \in E$ with $(e,f) = (e + if)$ [with $(e,0) \in E_{\mathbb{C}}$]

$$(x+iy)(e+if) = xe + ixf + iye - yf$$
$$= (xe - yf\,,\ i[xf+ye]) \Rightarrow (xe-yf\,,\ xf+ye)$$ (how happened)

introduce the $\mathbb{C}$-linear endomorphism $\sigma_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ by

$$\sigma_{\mathbb{C}}(e+if) = \sigma(e) + i\sigma(f)$$

from fund. thm of algebra we see $\sigma_{\mathbb{C}}$ has an eigenvalue $\lambda = a+ib$

a non-trivial eignvector $e+if$ with λ satisfy : $\sigma(e) + i\sigma(f) = (a+ib)(e+if)$

Thm let σ be an orthagonal transformation of the Euclidean vector space E of dimension n. $\exists$ a decomposition of E into an orthogonal sum of lines and planes stable under σ.

Pf with the lemma above we find linear subspace $R \subset E$ of dim $= 1$ or 2 ? stable under σ. It follows $R^{\perp}$ is stable under σ as well. Pf is about $\dim E$

=

Parabolic Forms

$\forall x \in X$

a finite dim. vector space X over $\mathbb{R}$ with positive quadratic form Θ $(\Theta(x) \geqslant 0)$

Cauchy-Schwarz Inequality : $\langle x,y\rangle^2 \leq \langle x,x\rangle\langle y,y\rangle$

Pf write in det form

$$\det\begin{bmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{bmatrix} \geqslant 0 \qquad \langle x,y\rangle \overset{?}{=} \langle y,x\rangle \ ?$$

$$x^2+y^2-2xy \to (x-y)^2 \geqslant 0 \ ?$$

trivial when x and y are linearly dependent ?

when x and y lin. independent we get from Gram matrix that the sign of det is unchanged if we replace x and y by orthanormal basis for the plane they span.

Corollary let (X,Θ) be positive quadratic form. A vector $\vec{x} \in X$ is isotropic iff $x \in X^\perp$ ie $\langle \vec{x},\vec{y}\rangle = 0$ $\forall x \in X$ doğru yazdım mı?

* (X,Θ) is parabolic if Θ is positive and the space $X^\perp$ is of dimension 1.

the isotropic line $X^\perp$ is invariant under the orthagonal transformations.

it follows we have a morphism of groups:

$$\mu : O(X) \to \mathbb{R}^*$$

where the multiplier $\mu(\sigma) \in \mathbb{R}^*$ denotes the eigenvalue of $\sigma \in O(X)$ on the isotropic line. This tells us to introduce a subgroup of $O(X)$ of index 2

$$O_\infty(X) = \{\sigma \in O(X) \mid \mu(\sigma) > 0\}$$

observe that the antipodal map $\bar{\iota}$, $x \mapsto -x$ belongs to centre of $O(X)$ and that $\mu(\bar{\iota}) = -1$. This defines a decomposition of the orthagonal group $O(X)$ of a parabolic space X : $O(X) = O_\infty(X) \times \mathbb{Z}/(2)$

The line at infinity parabolic setting useful for Euclidean Geometry

Let $\mathbb{E}$ be Euclidean space of dimension n and consider the following bilinear form on $X = \mathbb{E} \oplus \mathbb{R}$

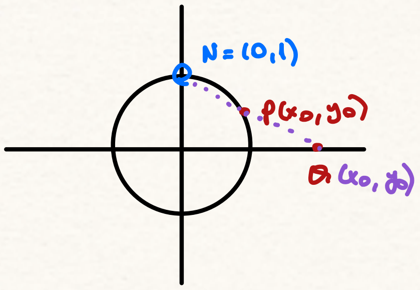
$$\langle (x,a),(y,b) \rangle = \langle x,y \rangle \qquad x,y \in \mathbb{E} \qquad a,b \in \mathbb{R}$$

it can be easily seen as parabolic. The isotropic line is generated by the vector $(0,1)$ and is called the "line at infinity" Now use inner product on X to introduce the "evaluation map" $ev((x,c),y) = \langle x,y \rangle + c : x,y \in \mathbb{E}, c \in \mathbb{R}$

the evaluation map identifies X with the space of affine functions on $\mathbb{E}$.

24.07 Lecture Notes

Stereographic Projection



$(y_0 < 1)$

$\varphi : S^1 \setminus \{N\} \to \mathbb{R}$ homeomorphism 1-1 + onto

$$m_{NP} = \frac{1-y_0}{0-x_0} = \frac{y_0-1}{x_0} \quad , \quad NP: y-1 = m(x-0) = \frac{y_0-1}{x_0} \cdot x \Rightarrow y = \left[\frac{y_0-1}{x_0}\right] x + 1$$

$$Q: PN \cap x\text{-axis} : y=0 \qquad -1 = \frac{y_0-1}{x_0} \cdot x \quad , \quad Q = \left(\frac{x_0}{1-y_0}, 0\right) ,$$

$$\varphi(x_0, y_0) = \frac{x_0}{1-y_0}$$

other direction : $\varphi \cdot \psi = id_{\mathbb{R}}$ $\quad \psi \cdot \varphi = id_{S \setminus \{N\}}$

$\psi : \mathbb{R} \to S \setminus \{N\}$ $\qquad Q(x_0, 0)$

$$m_{NQ} = \frac{0-1}{x_0-0} = \frac{-1}{x_0} \quad , \quad QN: y-1 = -\frac{1}{x_0}(x-0)$$

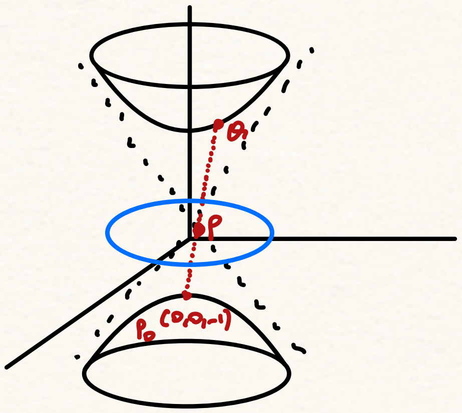
$QN \cap S^1$
$(x^2+y^2=1)$ $\quad , \quad y = \frac{-x}{x_0} + 1$

$$x^2+y^2 = x^2 + \left(\frac{-x}{x_0} + 1\right)^2 \Rightarrow x^2 + \frac{x^2}{x_0^2} - \frac{2x}{x_0} + 1 = 1$$

$$y = \frac{-2}{1+x_0^2} + 1 = \frac{x_0^2-1}{x_0^2+1} \qquad x\left[\left(1+\frac{1}{x_0^2}\right)x - \frac{2}{x_0}\right] = 0 \nearrow x = 0 \quad \searrow x = \frac{2/x_0}{1+\frac{1}{x_0^2}}$$

$$\psi(x_0) = \left(\frac{2x_0}{1+x_0^2}, \frac{x_0^2-1}{x_0^2+1}\right) \in S^1 \setminus \{N\}$$

for hyperboloid $\mathbb{R}^3$ $Q(x,y,z) = -x^2-y^2+z^2$ $S: -x^2-y^2+z^2=1$



$H^+ = S \cap \{z>0\}$

$H^- = S \cap \{z<0\}$

$D = \{(x,y,z) \in \mathbb{R}^3 \mid z=0 , x^2+y^2<1$

$\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = -x_1x_2 - y_1y_2 + z_1z_2$

noktayı vektör gör $\langle (x,y,z), (0,0,-1) \rangle = -z$

$P_0Q : (Q-P_0)t + P_0$ then $P_0Q \cap z=0 : \langle (Q - P_0 t + P_0), P_0 \rangle = 0$

solve : $t \cdot \langle Q - P_0, P_0 \rangle + \underset{1}{\overset{\shortparallel}{\langle P_0, P_0 \rangle}} = 0$

$t \cdot (\langle Q, P_0 \rangle + \underbrace{\langle -P_0, P_0 \rangle}_{-1}) + 1 = 0$

so $t = \dfrac{1}{t - \langle Q, P_0 \rangle}$ $(\langle Q, P_0 \rangle \leq -1) \Rightarrow t \geqslant 1/2$

$P = \dfrac{(Q-P_0)}{1-\langle Q, P_0 \rangle} + P_0$ OR $P = \dfrac{Q - P_0 \cdot \langle Q, P_0 \rangle}{1 - \langle Q, P_0 \rangle} = \dfrac{(x,y,z) - (-z)(0,0,-1)}{1-(-z)}$

$Q = (x,y,z)$ $P_0 = (0,0,-1)$ $P = \dfrac{(x,y,0)}{1+z} = \left(\dfrac{x}{1+z}, \dfrac{y}{1+z}, 0 \right)$

$\langle P_0, Q \rangle = -z$

Check : $\left(\dfrac{x}{1+z}\right)^2 + \left(\dfrac{y}{1+z}\right)^2 = \dfrac{x^2+y^2}{z^2+2z+1} = \dfrac{z^2-1}{z^2+2z+1} < 1$

disjoint union

H^+ and H^- are homeomorphic to $\underset{\downarrow}{D}$, $S = H^+ \sqcup H^-$ has 2 connected comp.

projeksiyon circle

a Lorentz Transformation is an element $\sigma \in O(\mathbb{R}^{n+1}, Q)$ st $\sigma(H^+) = H^+$

$$\begin{tikzcd}
 & \begin{array}{l} \sigma(H^+) = H^+ \\ \sigma(H^-) = H^- \end{array} \\
\sigma \in O(\mathbb{R}^{n+1}, Q) \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l} \sigma(H^+) = H^- \\ \sigma(H^-) = H^+ \end{array}
\end{tikzcd}$$

σ acts on S. $\langle x, y \rangle = \langle \sigma(x), \sigma(y) \rangle$

$S : \langle x, x \rangle = 1$ $\sigma : S \to S$ homeomorphism

ex $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ (antipodal map) $T \in O(\mathbb{R}^{n+1}, Q)$
$\quad x \to -x$

$T(H^+) = H^-$ $\quad T(H^-) = H^+$

lorentz transformations form a group under composition.

$Lor(\mathbb{R}^{n+1}, Q) \subset O(\mathbb{R}^{n+1}, Q)$

$$O(\mathbb{R}^{n+1}, Q) \cong Lor(\mathbb{R}^{n+1}, Q) \times \mathbb{Z}/2$$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad ||$

$\{id, T\}$ $\to$ antipodal map

T is involution $\quad T \neq id \quad T(T(\ )) = id \quad x \to -x \to x$

$[O(\mathbb{R}^{n+1}, Q) : Lor(\mathbb{R}^{n+1}, Q)] = 2$

Pick a vector $c \in \mathbb{R}^{n+1}$ with $\langle c, c \rangle = -1$ $\quad$ norm 1 hiperboloid $\quad$ norm -1 tek yapra hiperbo

reflection along c : $\quad T_c(x) = x + 2\langle x, c\rangle c$ $\quad T(x) = x - 2\dfrac{\langle x, n\rangle}{\underset{\hookrightarrow -1}{\langle n, n\rangle}} n$

$T_c \in O(\mathbb{R}^{n+1}, Q)$

Claim T_c is a Lorentz Transformation. $\quad$ Lor $H^+ \to H^+$, $H^- \to H^-$

pick $x \in S$, we must show that x and $\sigma(x)$ lie in the same component of S

$$\begin{aligned} \langle x, \sigma(x)\rangle &= \langle x, x + 2\langle x, c\rangle c\rangle \\ &= \langle x, x\rangle + 2\langle x, c\rangle\langle x, c\rangle = 1 + 2\langle x, c\rangle^2 > 0 \end{aligned}$$

$\det(T_c) = -1$ (reflection)

so Lorentz Transformation can have $\det = \mp 1$

$Lor^+(\mathbb{R}^{n+1}, Q) = \{\sigma \in Lor(\mathbb{R}^{n+1}, Q) \mid \det \sigma = 1\}$: even/special Lor. Trnsf.

$[Lor(\mathbb{R}^{n+1}, Q) : Lor^+(\mathbb{R}^{n+1}, Q)] = 2$

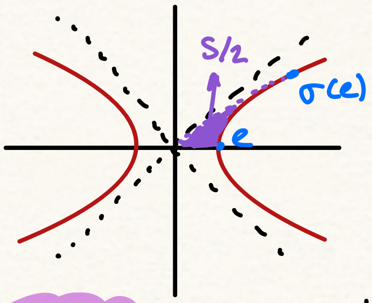
$$\begin{tikzcd} O(\mathbb{R}^{n+1}, Q) \arrow[d, dash, "2"'] & \to Q\text{yo koruyan} \\ Lor(\mathbb{R}^{n+1}, Q) \arrow[d, dash, "2"'] & \det = \mp 1 \quad \text{componentleri koruyor} \\ Lor^+(\mathbb{R}^{n+1}, Q) & \det = 1 \quad \text{üst component koruyor} \end{tikzcd}$$

$\mathbb{R}^2$ basis e, f $\langle e, e\rangle = 1$ $\langle f . f\rangle = 1$ $\langle e, f\rangle = 0$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = G$

$xe + tf \in S$ iff $\langle xe + tf, xe + tf\rangle = 1$

$\Rightarrow x^2 - t^2 = 1$ (hiperbol)

investigate Lorentz Transformations



suppose $\sigma \in \mathcal{L}or(\mathbb{R}^2, Q)$

$\sigma(e), e$ lie on same branch

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\pi \cdot \frac{s}{2\pi} = \frac{s}{2}$$

$\sigma(e) = (\cosh s)e + (\sinh s)f$, $\langle \sigma(f), \sigma(f)\rangle = -1 = \lambda^2 - \mu^2$

$\sigma(f) = \lambda e + \mu f$ $\langle \sigma(e), \sigma(f)\rangle = 0 = \lambda \cosh s - \mu \sinh s$

$\sigma(f) = \mp[(\sinh s)e + (\cosh s)f]$

$\downarrow$

$\text{sign}(t)$

$$\sigma(f) = \begin{bmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{bmatrix} \longrightarrow \det = 1 \ (+\ \text{sign})$$

, $\det(\sigma) = 1$

$\mathcal{L}(s+t) = \mathcal{L}(s) . \mathcal{L}(t)$

$L(s) L(t) = \mathcal{L}(s+t)$ proof

$$\mathcal{L}(s) . \mathcal{L}(t) = \underbrace{\begin{bmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{bmatrix}}_{\mathcal{L}(s)} \underbrace{\begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}}_{\mathcal{L}(t)} = \begin{bmatrix} cs . ct + ss\, st & cs\, st + ss\, ct \\ ss\, ct + cs\, st & ss\, st + cs\, ct \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cosh(s+t) & \sinh(s+t) \\ \sinh(s+t) & \cosh(s+t) \end{bmatrix}}_{\mathcal{L}(s+t)}$$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Sign (-) : $\sigma(f) = -[(\sinh s)\, e + (\cosh s)\, f]$

$$\sigma = \begin{bmatrix} \cosh(s) & -\sinh(s) \\ \sinh(s) & -\cosh(s) \end{bmatrix} \qquad \det(\sigma) = -1 \,, \quad \mathrm{tr}(\sigma) = 0 \quad \Rightarrow \text{reflection}$$

→ eigenvalue ∓ 1

(-) sign $\sigma(f)$ matrix hali → $\det = -1$ (-sign)

$\mathrm{Lor}^{+}(\mathbb{R}^2, \mathcal{Q}) \cong (\mathbb{R}, +)$

Thm Any $\sigma \in \mathrm{Lor}(\mathbb{R}^{n+1}, \mathcal{Q})$ is the product of at most $n+1$ reflections of the form T_c, $\langle c, c\rangle = 1$
-1

Möbius Transformations

E: Euclidean Vector Space $\dim E = n$ $(-0, n)$ -1 yok n tane 1 var

C: sphere with center c, radius $r>0$

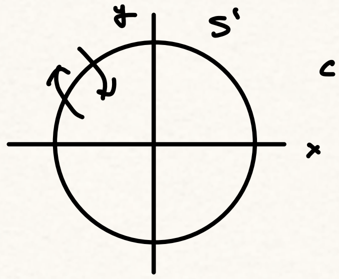
$C = \{ x \in E \mid \mathcal{Q}(x-c) = r^2 \}$

inversion in the sphere C : $\sigma : E - \{c\} \to E - \{c\}$ $\mathbb{R}^3$

Möb $\left(x \to c + r^2 \cdot \frac{|x-c|}{|x-c|^2} \right)$ OR $\left(c + r^2 \frac{x-c}{\mathcal{Q}(x-c)} \right)$

inversion içi dışına dışı içine

ex



$c = 0$, $r = 1$

$|(x,y)| \cdot |\sigma(x,y)| = 1$

$\sigma : x \to \frac{x}{|x|^2}$

$\sigma : (x,y) \to \frac{(x,y)}{x^2+y^2}$ → $\frac{\sqrt{x^2+y^2}}{x^2+y^2}(x,y) = 1$

$\sigma(\sigma(\,)) = x.y$, σ is an involution → σ not id but σ^2 id

$\sigma(\sigma(\,)) = \sigma$

involution

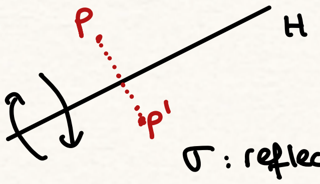
if $x \in C$ then $\sigma(x) = x$ kürenin üstünde kalanlar etkilenmiyor

$\hat{E}$: 1-pt compactification $\hat{E} = E \cup \{\infty\}$

$\sigma : \hat{E} \to \hat{E}$
$c \to \infty$
$\infty \to c$

with a suitable topology (open/close set) on E σ is homeomorphism. $(\sigma^{-1} = \sigma)$

$H \subset E$ affine hyperplane (affine orijinden geçmek zorunda değil demek)



σ : reflection in H (H orijinden gecmezse liner değil)

So σ is involution, homeomorphism

sphere in $\hat{E}$:
- euclidean sphere
- affine hyperplane in $E + \{\infty\}$

inversion in a sphere in $\hat{E}$
- inversion in a Euclidean sphere
- refl along hyperplane

Homeo$(\hat{E})$ is a group under composition

Möb(E) is the subgroup of Homeo$(\hat{E})$ gen. by inversions in spheres in $\hat{E}$.

$E \equiv \mathbb{R}^n$ euclidean

Möb$(E) \subset$ Homeo$(\hat{E})$

$E \oplus \mathbb{R}^2$ $\quad \dim(E \oplus \mathbb{R}^2) = n+2$ $\quad e, f \in E$ $\quad a,b,c,d \in \mathbb{R}$

$$\langle (e,a,b),(f,c,d)\rangle = \underbrace{\underbrace{-\langle e,f\rangle}_{\text{Syl. type}} + \underbrace{\frac{1}{2}(ad+bc)}_{(-1,1)}}_{\text{Sylv. Type } (-n,-1,1)}$$

$\mathbb{R}^2 : \langle (a,b),(c,d)\rangle = \frac{1}{2}(ad+bc)$, $\Theta(1,1) = 1$ $\quad x_1 = (1,1)$

$(\mathbb{R}x_1)^\perp = \{(a,b) \in \mathbb{R}^2 \mid \langle (a,b),(1,1)\rangle = 0\}$

$$\langle (a,b),(1,1)\rangle \overset{\|}{\;} \tfrac{1}{2}(a+b) = 0$$

Lor$(E \oplus \mathbb{R}^2, \Theta) \cong$ Möb(E)

T_c , $\langle c,c\rangle = -1$ ⟷ inversions in sphere in $\hat{E}$

odd/even Möbius tr.

$x_2 = (1,-1) \in (\mathbb{R}x_1)^\perp$

$$\begin{aligned}\Theta(x_2) &= \langle x_2, x_2\rangle \\ &= \langle (1,-1),(1,-1)\rangle \\ &= \tfrac{1}{2}(-1-1) = -1\end{aligned}$$

x_1, x_2 : gram matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$