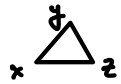
MT-2 Subjects 29.07 Pazartesi

Metric Spaces

1) $X \neq \emptyset$
2) $d: X \times X \to \mathbb{R}$
3) $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x=y$
4) $d(x,y) = d(y,x)$

*5) $d(x,z) \leq d(x,y) + d(y,z)$



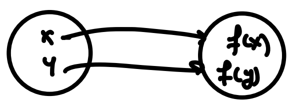
(x,d) ama d'yi yazmasan da dır

Defn X, Y metric spaces, $f: X \to Y$ isometry if f is bijective and

$$d_Y(f(x), f(y)) = d_X(x,y)$$

$d_X \to$ X'i yazmana gerek yok.

$x, y \in X$



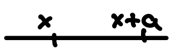
isometry $X \to X$ form a group under composition $\mathrm{Isom}(X)$

$\mathbb{R}$, $d(x,y) = |x-y|$ $\left(\sqrt{(x-y)}\right)^2$?

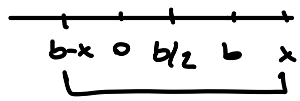
Proposition an isometry $\sigma: \mathbb{R} \to \mathbb{R}$ is either translation ($x \to x+a$) or reflection ($x \mapsto b-x$)

isometry — translation; isometry, involution — reflection

$x \to y$
$x \to y \to z$



$0 \to b$
$b \to 0$



$x \to b-x \to b-(b-x) = x$
involution

Pf by contr. let T be the reflection or translation which agrees with σ at 0 or 1

$\sigma(0) \neq \sigma(1)$

$\sigma(1) > \sigma(0)$
$\sigma(1) = \sigma(0)+1$

$T(x) = x + \sigma(0)$
$T(0) = \sigma(0)$
$T(1) = \sigma(0)+1$
$= \sigma(1)$

translation

$\sigma(1) < \sigma(0)$
$\sigma(1) = \sigma(0) - 1$
$T(x) = \sigma(0) - x$

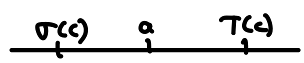
$T(0) = \sigma(0)$
$T(1) = \sigma(0) - 1 = \sigma(1)$

reflection

Claim $\sigma = T$ Suppose $\exists c \in \mathbb{R}$ st $\sigma(c) \neq T(c)$

$a = \sigma(0) = T(0)$
$b = \sigma(1) = T(1)$

$$d(\sigma(c), a) = d(\sigma(c), \sigma(0)) = d(c,0)$$
$$= d(T(c), T(0)) = d(T(c), a)$$



a is the midpoint of $\sigma(c)$, $T(c)$
b " " " " " "
$a = b \Rightarrow$ contradiction $\square$

Isom(IR) is gen by translations and $\iota: x \to -x$

$f(x) = b - x$ (reflection) $\qquad x \xrightarrow[\text{refl.}]{\iota} -x \xrightarrow[\text{transl.}]{g} -x + b$

$b - x = g(\iota(x))$

Trans (IR) $\subset$ Isom (IR) is a $\overset{\text{normal}}{\text{subgroup}}$

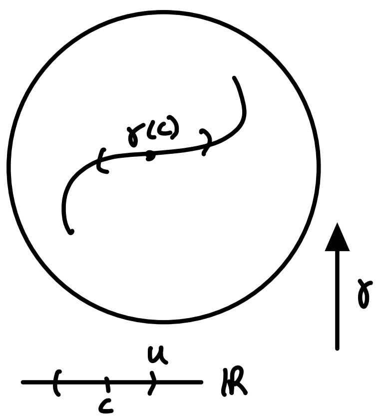
$f(x) = x + a_1$
$g(x) = x + a_2$ $\quad > \quad f \circ g(x) = x + (a_1 + a_2) \qquad$ Trans (IR) $\cong$ IR $\to$ under (+)

for normality $\quad$ Isom (IR) / Trans (IR) $\cong \mathbb{Z}/2 \qquad$ Isom(IR) is the semi-direct product of IR and $\mathbb{Z}/2$

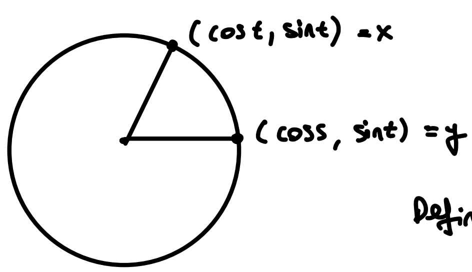
$\mathbb{Z}/2$ is isometric to $O(1)$
$O(\mathbb{R}^n, Q)$, $Q(x) = x^2$

Defn *** $J \subseteq \mathbb{R}$ is interval (open/close), X: metric space

A curve $\gamma: J \to X$ is a geodesic if each pt $c \in J$ has a neighbor-hood $u \subset J$ s.t. $\gamma|_u : u \to X$ is a distance-preserving.



ex $X = S^1 \qquad x, y$ is not coordinate product



Define $d: S^1 \times S^1 \to \mathbb{R}$
$d(x, y) = \alpha \in [0, \pi]$

$\gamma: \mathbb{R} \to S^1$
$\gamma(t) = (\cos t, \sin t)$ is a geodesic.

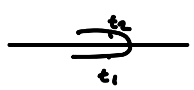
ex devam
if $s, t \in (-\pi, \pi)$ then $\alpha = |s - t|$

γ is locally distance preserving.

lemma $\mu \in J \subseteq \mathbb{R}$ open interval

$\gamma: J \to \mathbb{R}$ is a geodesic with $\gamma(\mu) = 0$ then $\exists\, \varepsilon \in \{+1, -1\}$ s.t. $\gamma(t) = \varepsilon(t - \mu)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t - \mu \qquad \mu - t$



yön değiştirme yok geodesicte

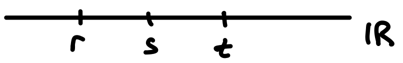
$\gamma(t_1) = \gamma(t_2)$ but $t_1 \neq t_2$

$\mathbb{R}^n$, $Q = x_1^2 + \dots + x_n^2$

proposition a geodesic curve $\gamma : \mathbb{R} \to E$ has the form $\gamma(t) = et + v$ where

$v \in E$ and $e \in E$, $|e| = 1$

E (euclidean space) ise geodesic ler  doğru olur.

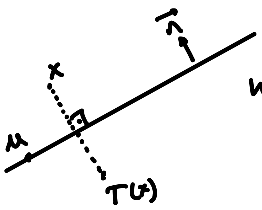


$|r-t| = |r-s| + |s-t|$

$|\gamma(r) - \gamma(t)| = |\gamma(r) - \gamma(s)| + |\gamma(s) - \gamma(t)|$

locally distance preserving $\gamma : \mathbb{R} \to S^1$
↳ $\mathbb{R}^1$ deki noktaları S^1'ye transfer ediyor.

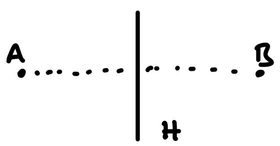
$E = \mathbb{R}^n$ (euclidean vector space), Isometries of E



H affine hyperplane $\quad |n| = 1 \quad \vec{n} \perp H$

$T(x) = x - 2\langle x, u \rangle n$: reflection along H

H perpendicularly bisects the line segment $x\,T(x)$

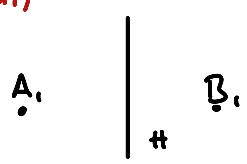


$\{x \in E \quad d(x, A) = d(x, B)$

lemma $A_1, \dots, A_p$ and $B_1, \dots, B_p$ of points of E st $d(A_i, A_j) = d(B_i, B_j)$

then $\exists$ isometry σ of E composed of at most p reflections st $\sigma(A_i) = B_i$

pf (induction)

$p = 1$ 

$T_H(A_1) = B_1 \quad T_H(B_1) = A_1 \quad A_1 = B_1$ use $\sigma = id$

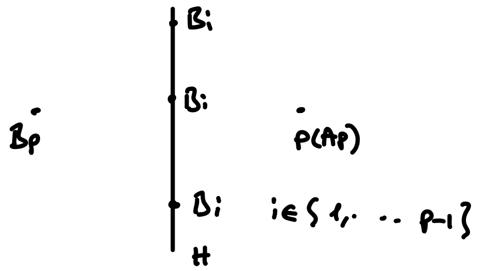
$\sigma = T_H$

suppose $\exists$ an isometry ρ, composed of $\leq p-1$ reflections, s.t. $\rho(A_i) = B_i$, $i = 1, \dots, p-1$

case 1: $\rho(A_p) = B_p$ take $\sigma = \rho$

case 2: $\rho(A_p) \neq B_p$, $d(\rho(A_p), B_i) = d(\rho(A_p), \rho(A_i))$ for i $1 \dots p-1$

$d(A_p, A_i) = d(B_p, B_i)$



take $\sigma = T_H \cdot p$

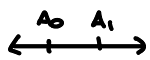
$\sigma(A_i) = T_H(p(A_i)) = T_H(B_i) = B_i$ $\quad i = 1 \dots, p-1$

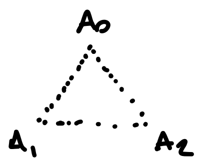
$\sigma(A_p) = T_H(p(A_p)) = B_p$ $\quad \square$

$B_1, \dots B_{p-1}$ is line on H. $\quad$ (n boyutlu $n+1$ reflection)

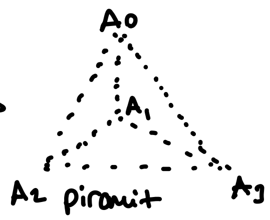
Euclidean simplex ☆ ($E \cong \mathbb{R}^n$)

$A_0, \dots, A_n$ of points of $E\ (\cong \mathbb{R}^n)$ NOT contained in an affine hyperplane $\quad \overset{?}{(n+1)}$

$n=1$  $E \cong \mathbb{R}^1$ $\quad A_0 \neq A_1 \quad A_1 - A_0 \neq 0$ (non-zero vector)

$n=2$ 

$A_1 - A_0, \ A_2 - A_0$ lin. indep.

$n=3$ 

$A_1 - A_0, \ A_2 - A_0, \ A_3 - A_0$ lin. indep.

$A_0, \dots A_n$ form a Euclidean Simplex in $E \cong \mathbb{R}^n$ iff $A_1 - A_0 \dots, A_n - A_0$ are lin. indep.

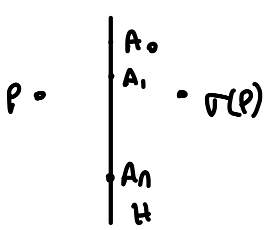
lemma ☆ $A_0, \dots, A_n$ euclidean simplex if two isometries $\alpha, \beta \in E$ agree on $A_0, \dots A_n$, then $\alpha = \beta$

pf $\sigma = \beta^{-1}\alpha$ (is identity)

suppose $\exists$ a $P \in E$ st $\sigma(P) \neq P$ need contradiction

$$d(\sigma(P), A_i) = d(\sigma(P), \sigma(A_i)) = d(P, A_i)$$

$\alpha(A_i) = \beta(A_i)$, $\ \beta^{-1}\alpha(A_i) = A_i$



for $i = 0, \dots n$ contr. $\square$

$A_0, A_1 \dots$ bulunmayacak dedik ama bulunuyorlar

Corollary σ isometry fixes an affine hyperplane K pointwise, then either $\sigma = id$ or $\sigma = T_k$

(Pf)?

Thm $\beta \in Isom(\mathbb{R}^n)$ then β can be written as the product of $\leq n+1$ reflections.

$n=1$ $\quad \underset{\text{refl.}}{x \to b-x}$ or $\underset{\text{transl.}}{x \to x+a}$ en fazla 2 refl. olarak yazabilirsin diyor thm.

$f_1(x) = b_1 - x \qquad f_2 = b_2 - x \qquad f_1 \circ f_2(x) = b_1 - (b_2 - x) = x + b_1 - b_2 \qquad b_1 - b_2 = a$

her ötelene 2 reflection composition şeklinde yazılabiliyor.

Pf pick a Euclidean Simplex $A_0, \dots A_n$ use lemma $\sigma(A_i) = B_i$ to produce σ,

composed of $\leq n+1$ refl. st. $\sigma(A_i) = \beta(A_i) \qquad (A_i = A_i \ , \ \beta = \beta(A_i))$

Then $\sigma = \beta$ $\quad \square$

isometriler reflectionlar tarafından generate ediliyor. *

$e \in E$, $T_e(x) = x + e$ translation

Translations form a subgroup $T(E) \subset Isom(E)$, $T(E) \cong E \cong \underset{\text{under } (+)}{(\mathbb{R}^n)}$ $\quad T_e \circ T_f = T_{e+f}$

$\rho \in Isom(E)$, $\rho(0) = e$

$\rho = T_e \circ \sigma \qquad \sigma \in O(e)$ [orthogonal transformation ... her isometriyi ....]

$\sigma = \vec{\rho}$: linearization

$Isom(E) / T(E) \cong O(E)$

normal subg.

ötelene subgroupu izometri içinde normal, quotienti da $O(E)$

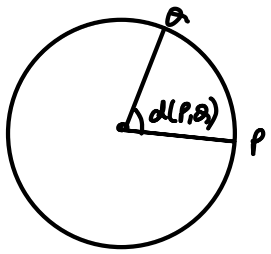
Spheres

$F = \mathbb{R}^{n+1}$ Euclidean

$S(F) = S^n = \{ x \in F \mid \langle x, x \rangle = 1 \}$

$P, Q \in S^n$ $\quad |\langle P, Q \rangle| \leq |P| \cdot |Q|$ → vektörün normunun karesidir

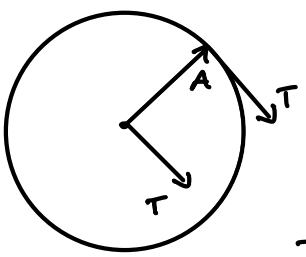
$|\langle P, Q \rangle| \in [-1, 1]$ $\quad \cos d(P,Q) = \langle P, Q \rangle$ $\quad d(P,Q) \in [0, \pi]$
↳ spherical distance



$d(P,Q) \geq 0$ $\quad = 0$ iff $P = Q$ $\quad$ inner prod $= -1$ $(\pi - 2\pi)$

$d(P,Q) = d(Q,P)$ $\quad$ trig ineq ✓ but will prove later.

Defn $A \in S^n$ a tangent vector to S^n at A is a vector $T \in F$ with $\langle T, A \rangle = 0 \Rightarrow T \perp A$



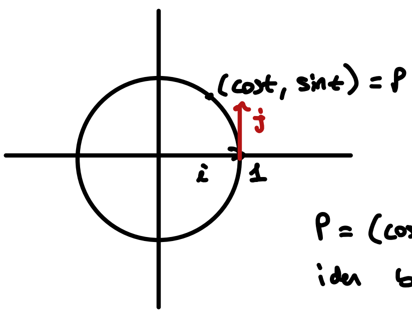
$T_A(S^n)$: vector space of tangent vectors to S^n at A.

Sylvester type $(0, n+1)$

$T_A(S^n) = (\mathbb{R} \cdot A)^\perp \Rightarrow F = \overset{(0,1)}{\mathbb{R} \cdot A} \oplus \overset{(0,n)}{(\mathbb{R} \cdot A)^\perp}$

$$G = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ 0 & \boxed{\begin{matrix} n \times n \\ I \end{matrix}} \\ \vdots & \\ 0 & \end{bmatrix}$$

n+1 tane 1

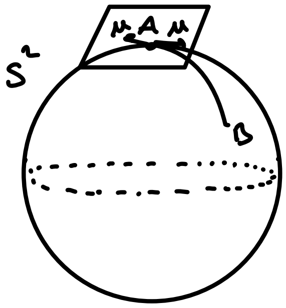


$P = (\cos t) i + (\sin t) j$: unit tangent vector to the pt. on S^1
i'den başla j'den git

Lemma A, B pt in S^n. A unit tangent vector u to S^n at A can be chosen st

& $B = \cos d(A,B) \, A + \sin d(A,B) \, u$ $\quad$ (A'dan B'ye nasıl gitme formülü)

Pf suppose A, B lin indep. let $u \in T_A(S^n)$ $\quad |u| = 1$, $u \in$ plane span by A, B
↳ teğet uzayı boyutu n (küreyi düzlemle kes 1 boyutlu çember)

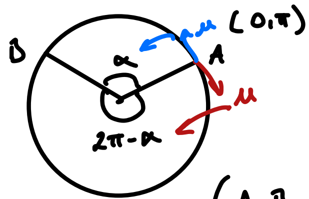


$B = x.A + y.\mu \qquad (\mu \in \underbrace{R \cap T_A(S^n)}_{\dim = 1})$

$\langle B, B\rangle = 1 = \langle xA + y\mu\, ,\, xA + y\mu\rangle$

$= x^2 + y^2$ (cost sint formülüne geliyor)

choose $s \in [-\pi, \pi]$ st $B = \cos s\, A + \sin s\, \mu$. Assume WLOG $S \in [0, \pi]$

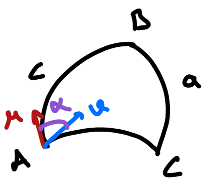


then $d(A,B) = \langle A, B\rangle$

$= \langle A, \cos A + \sin \mu\rangle = \cos S \Rightarrow S = d(A,B)$ $\square$

(A, B lin indep $\quad B = A \quad s = 0$, $\underbrace{B = -A}_{\text{kutuplar}}$ $\quad S = \pi$ take any μ)

let A, B, C be pt on sphere (lin indep)



$a = d(B,C) \in (0,\pi)$
$b = d(A,C) \in (0,\pi)$
$c = d(A,B) \in (0,\pi)$

$B = \cos c\ A + \sin c\ \mu$
$C = \cos b\ A + \sin b\ \nu$
$\cos\alpha = \langle \mu, \nu\rangle$

[spherical triangle]

think α as classic angle of triangle

Cosine relation $\quad \cos a = \cos b \cos c + \sin b \sin c \cos \alpha$

$$\begin{vmatrix} \langle A,A\rangle & \langle B,A\rangle \\ \langle A,C\rangle & \langle B,C\rangle \end{vmatrix} = \begin{vmatrix} \langle A,A\rangle & \cos c\langle A,A\rangle + \sin c\langle A,\mu\rangle \\ \langle A,C\rangle & \cos c\langle A,C\rangle + \sin c\langle C,\mu\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle A,A\rangle & \sin c\langle A,\mu\rangle \\ \langle A,C\rangle & \sin c\langle C,\mu\rangle \end{vmatrix} = \begin{vmatrix} \langle A,A\rangle & \sin c\langle A,\mu\rangle \\ \cos b\langle A,A\rangle + \sin b\langle A,\nu\rangle & \sin c.(\cos b\langle A,\mu\rangle + \sin b\langle \mu,\nu\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle A,A\rangle & \sin c\langle A,\mu\rangle \\ \sin b\langle A,\nu\rangle & \sin c \sin b\langle \mu,\nu\rangle \end{vmatrix} = \sin b \sin c \begin{vmatrix} \langle A,A\rangle & \langle A,\mu\rangle \\ \langle A,\nu\rangle & \langle \mu,\nu\rangle \end{vmatrix} = \begin{vmatrix} \langle A,A\rangle & \langle B,A\rangle \\ \langle A,C\rangle & \langle B,C\rangle \end{vmatrix}$$

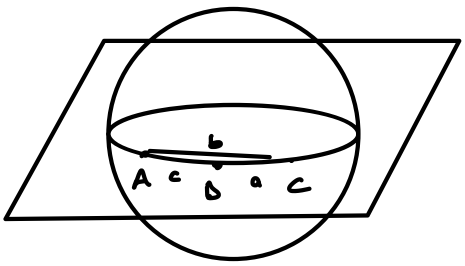
$$\Rightarrow \sin b \sin c \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \begin{vmatrix} 1 & \cos c \\ \cos b & \cos a \end{vmatrix}$$

$\Rightarrow \cos a - \cos b \cos c$ Mclaren series

$$1 - \frac{a^2}{2} = \left(1 - \frac{b^2}{2}\right)\left(1 - \frac{c^2}{2}\right) + \cos\alpha\,(b - \dots)(c - \dots)$$

$$\frac{a^2}{2} = \frac{b^2}{2} + \frac{c^2}{2} - \cos\alpha\, b \cdot c \quad \Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos\alpha$$

if A, B, C not lin indep

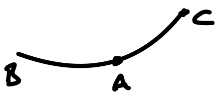


$\alpha = 0$: $\cos a = \cos b \cos c + \sin b \sin c = \cos(b-c) = \cos(c-b)$

or

$\alpha = \pi$: $\cos a = \cos b \cos c - \sin b \sin c = \cos(b+c)$

$(a = b + c)$



Triangle Inequality say A, B, C lin indep $\alpha \in (0, \pi)$ $\cos\alpha < 1$

$\cos a = \cos b \cos c + \cos\alpha \sin b \sin c$

$< \cos b \cos c + \sin b \sin c$

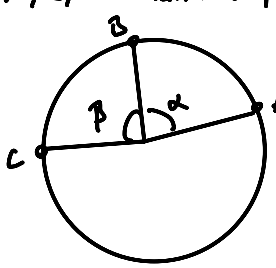
$\quad\; \| $

$\cos(b-c) = \cos(c-b)$

if $c > b$: $a > c - b \Rightarrow a + b > c$ is trivial

$c < b$: $a > b - c \Rightarrow a + c > b$ (permutation pf)

A, B, C lin. dep.



$\alpha + \beta \leq \pi$

B between A, C

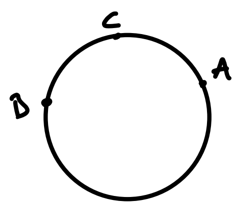
$\Rightarrow \alpha \leq \alpha + \beta + \beta$

spherical distance is a metric on S^n.

min   max

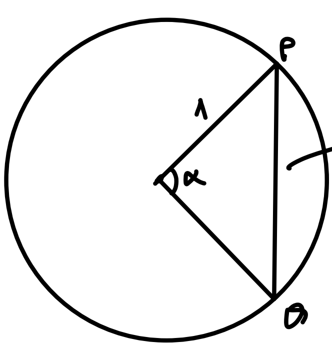
$\pi < \dots < 3\pi$

triangle   triangle



$d(A, B) = d(A, C) + d(C, B)$

the two metrics $|P - Q|$ and $d(P, Q)$ on S^n define the same topology

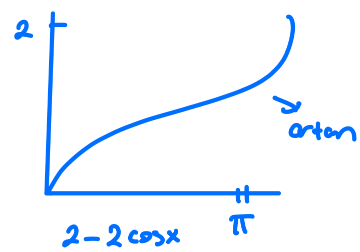


$|P - Q|$

$|P - Q|^2 = 1 + 1 - 2\cos\alpha$

$|P - Q| = \sqrt{2 - 2\cos d(P, Q)}$

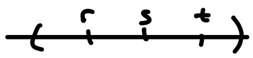
look at $2 - 2\cos x$ fn graph



Proposition $\gamma : \mathbb{R} \to S^n$ geodesic

$$\gamma(t) = A.\cos t + T.\sin t \qquad A \in S^n \quad (P+)$$

T: unit tangent vector to S^n at A.

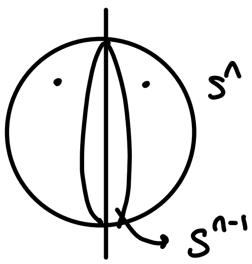


$$|r-t| = |r-s| + |s-t|$$
$$\Downarrow$$
$$d(\gamma(r), \gamma(t)) = d(\gamma(r), \gamma(s)) + d(\gamma(s), \gamma(t)) \Rightarrow \gamma(r)\ \gamma(s)\ \gamma(t) \text{ are lin. indep.}$$

spherical distance metric

Lemma $A_1 \dots A_p$, $B_1 \dots B_p \in S^n$ st $d(A_i, A_j) = d(B_i, B_j)$
$[\text{iff } |A_i - A_j| = |B_i - B_j|]$

Then $\exists\ \sigma$ composed of at most p hyperplane reflections st $\sigma(A_i) = B_i$



Thm any $\sigma \in \mathrm{Isom}(S^n)$ can be obtained from restrictions of an element of $O(n+1)$ to S^n

$$O(n+1) = O(\mathbb{R}^{n+1})$$

orthogonal group S^n: kuror

$$\mathrm{Isom}(S^n) \cong O(n+1) = O(\mathbb{R}^{n+1})$$

$$\mathrm{Isom}(S^1) \cong O(2) \cong \mathbb{R}/2\pi\mathbb{Z}$$
$$O(2) \cong S^1$$

31. 07 Çarşamba

Hyperbolic Spaces (2d Hyperbolic Plane)

$F \cong \mathbb{R}^{n+1}$ $\quad \Theta : (-n, 1)$ $\qquad S(F) = \{x \in F \mid \langle x, x \rangle = 1\}$

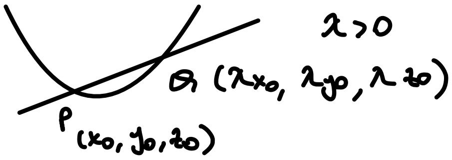
H^n : the sheet with $x_n > 0$

$P, Q \in H^n \Rightarrow \langle P, Q \rangle \geq 1$ $\quad (|\langle P, Q \rangle| \geq 1$, $\quad P, Q \in S(F)$ are in same sheet iff $\langle P, Q \rangle > 0$

$\cosh d(P,Q) = \langle P, Q \rangle$ $\qquad d : H^n \times H^n \to \mathbb{R}$ $\qquad d(P,Q) \geq 0$,

$d(P,Q) = 0$ iff $\langle P, Q \rangle = 1$ iff $P = Q$

$d(P,Q) = d(Q,P)$ simetrik



$\lambda > 0$

$$-x^2 - y^2 + z^2 = 1$$
$$-x_0^2 - y_0^2 + z_0^2 = 1$$

$$\lambda^2(-x_0^2 - y_0^2 + z_0^2) = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = 1$$

$\cosh(-x) = \cosh(x) \quad * \quad \dfrac{e^x + e^{-x}}{2}$

Triangle Inequality $A, B, C \in H^n \Rightarrow d(A,B) \le d(A,C) + d(B,C)$ [sharp if lin indep]

Pf $a = d(B,C) \quad b = d(A,C) \quad c = d(A,B)$, assume A, B, C lin ind.

$R = \langle A, B, C \rangle \qquad \dim R = 3$

$(-2, 1)$

$2 + 1 = 3$

$R \oplus R^\perp \quad$ n tane $-1 \quad$ 1 tane 1

$(-2, 1) \quad (-n+2, 0)$

$-(n-2)$

4 $n-2$ tane -1 var.

(non singular)

$$\Delta = \begin{array}{c|ccc|c} & & & & \\ & 1 & \cosh a & \cosh b & c \\ & \cosh a & 1 & \cosh c & b \\ & \cosh b & \cosh c & 1 & a \end{array}$$

$$\Delta = 1 \cdot 1 - \cosh^2 c - \cosh a(\cosh a - \cosh b \cosh c) + \cosh b(\cosh a \cosh c - \cosh b)$$

$$= 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c$$

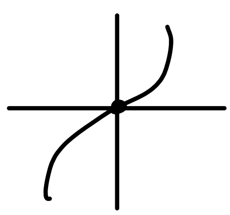
$$= (\cosh^2 b - 1)(\cosh^2 c - 1) - (\cosh b \cosh c - \cosh a)^2 =$$

remember $\cosh^2 x - \sinh^2 x = 1$

$$= \sinh^2 b \sinh^2 c - (\cosh b \cosh c - \cosh a)^2$$

$$= (\cosh a - \cosh b \cosh c + \sinh b \sinh c) \cdot (\cosh b \cosh c - \cosh a + \sinh b \sinh c)$$

remember $\cos(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y$



$p = \dfrac{a+b+c}{2}$ → özgen çevre

$$= (\cosh a - \cosh(b-c)) \cdot (\cosh(c+b) - \cosh a)$$

remember $\cosh x - \cosh y = 2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$

$$= 4 \sinh\left(\frac{a+b-c}{2}\right) \sinh\left(\frac{a-b+c}{2}\right) \cdot \sin\left(\frac{a+b+c}{2}\right) \sinh\left(\frac{c+b-a}{2}\right) = \Delta = 4 \sinh p \cdot \sinh(p-a) \sinh(p-b) \sinh(p-c)$$

$\Delta = \underset{+}{4} \, \underset{+}{\underline{\sinh p}} \cdot \sinh(p-a) \sinh(p-b) \sinh(p-c)$ $\quad \Delta > 0 \quad p = \dfrac{a+b+c}{2} > 0$

4 Syl. Type $[-1,-1,1]$ det >0

if $c \geq a$, $c \geq b$ then : $p - c > 0$

$\Downarrow$

$p-a = \dfrac{b+c-a}{2} > 0$, $p-b = \dfrac{c+a-b}{2} > 0$

then $a+b-c = 2.(p-c) > 0 \Rightarrow c < a+b$ $\square$

so $d(P,Q)$ defines a metric on H^n.

Defn $A \in H^n$, a tangent vector to H^n at A is a $T \in F$ s.t. $\langle A, T \rangle = 0$

$T_A(H^n) = (\mathbb{R}.A)^\perp$ is the tangent space to H^n at a

note : $\underset{n+1}{F} = \underset{\underset{1}{(0,1)}}{(\mathbb{R}.A)} \oplus \underset{\underset{n}{(-n,0)}}{(\mathbb{R}.A)^\perp}$ so $T_A(H^n)$ sylv. type $(-n,0)$ $\quad$ dim $T_A(H^n) = n$

negative definite

teğet vektörlerin kendileriyle inner productları negative $\quad (-x_1^2 - \dots - x_n^2 \;\; ?)$

if $\langle T, T \rangle = -1$ then T is a unit tangent vector.

Lemma $A, B \in H^n$, it's possible to choose a unit tangent vector $\mu \in T_A(H^n)$ st

$B = A . \cosh d(A,B) + \mu \sinh d(A,B)$ $\quad$ eştbörle, kareyi de eştbörle

Pf if $A = B$, pick any unit tangent vector $\mu \in T_A(H^n)$

Suppose $A \neq B$, $R = \langle A, B \rangle$ has Sylv Type $(-1,1)$ [discriminant lemma $\overset{\text{norm } 1}{(-1,1)}$ $\cancel{(-1,0)}$ $\cancel{(-1,0)}$]

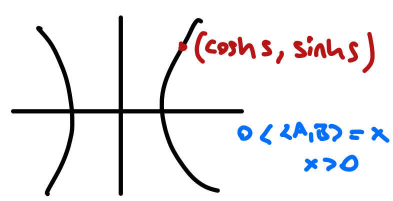
$R = \underset{(0,1)}{\mathbb{R}.A} \oplus \underset{(-1,0)}{(\mathbb{R}.A)^\perp}$ $\quad$ dim $(\mathbb{R}.A)^\perp = 1$ $\quad (-1,0)$

pick $\mu \perp A$, $\langle \mu, \mu \rangle = -1$ $\quad \mu \in R$

$\{A, \mu\}$ form an orthonormal basis for $R\}$ $\Rightarrow B = x.A + y.\mu$

$1 = \langle B, B \rangle = x^2 \underbrace{\langle A, A \rangle}_{1} + 2xy \underbrace{\langle A, \mu \rangle}_{0} + y^2 \underbrace{\langle \mu, \mu \rangle}_{-1} \Rightarrow x^2 - y^2 = 1$ (hiperbol üstünde)

$B = A . \cosh s + \mu \sinh s$ (s herhangi bir parametre) $\quad (s \geq 0)$



$\cosh d(A,B) = \langle A,B\rangle = \cosh s \Rightarrow s = d(A,B)$

$\Rightarrow B = A\cosh d(A,B) + u\sinh d(A,B)$ 1 noktadan diğer noktaya nasıl geçerim?

geodesics

$\gamma : \mathbb{R}\to H^n$ $\gamma(t) = A\cosh t + T\sinh t$ $A\in H^n$ $A = \gamma(0)$

unit tangent vector

$T = \gamma'(0)$ $T\in T_A(H^n)$ $\langle T,T\rangle = 1$

kürede global parametrizasyon korunmuyor ama hiperbolde korunuyor (euclidde de korunuyor)

$\langle\gamma(t),\gamma(s)\rangle = \langle A\cdot\cosh t + T\sinh t, A\cosh s + T\cosh s\rangle$

$= \cosh t\cosh s - \sinh t\sinh s$

assume $t\geq s$ $= \cos(t-s)$

$\cosh d(\gamma(t),\gamma(s)) = \langle\gamma(t),\gamma(s)\rangle$

$d(\gamma(s),\gamma(t)) = |t-s|$

note: geodesics are globally distance-preserving

hyperbolic line: image of a geodesic curve.

through two distinct pts in H^n there posses a unique hyp. line

isometries

$Lor(F) = \{\sigma\in O(F) \mid \sigma(H^n) = H^n\}$ $[O(F) : Lor(F)] = 2$

$Lor(F)$ is generated by Lorentz reflections $T_n = x + 2\langle x,n\rangle n$, $\langle n,n\rangle = 1$

Thm an isometry $\beta\in Isom(H^n)$ is induced by a Lorentz Transformation of F.

meaning $Isom(H^n)\cong Lor(F)$ [metrik tanımla]

Poincaré-Disk Model / Stereographic Projection model $(-n,0)$ $(0,1)$

$E\cong\mathbb{R}^n$ euclidean $(0,n)$ $F = E\oplus\mathbb{R}$ (orthogonal direct sum) $\langle(A,a),(B,b)\rangle = -\langle A,B\rangle + ab$

has Sylv. Type $(-n,1)$

üst yaprak

$H^n = \{(A,a)\in F \mid -\langle A,A\rangle + a^2 = 1, a>0\}$

$H^n = \{ (A,a) \in F \mid -\langle A,A\rangle + a^2 = 1 \;,\; a>0 \}$ üst yaprak

$D^n = \{ A \in E \mid \langle A,A\rangle < 1 \}$

$f: D^n \to H^n \qquad f(P) = \dfrac{(2P,\, 1+\langle P,P\rangle)}{1-\langle P,P\rangle}$ stereographic proj. with center $(0,-1)$

$$\cosh d(P,Q) = 1 + 2\cdot\frac{|P-Q|^2}{(1-|P|^2)(1-|Q|^2)} \qquad P,Q \in D^n$$

this defines a metric on D^n
it is known as Poincare's Disk

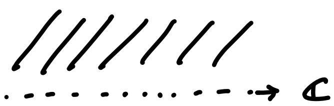
$$\cosh d(P,Q) = \left\langle \frac{(2P,\, 1+\langle P,P\rangle)}{1-|P|^2} ,\ \frac{(2Q,\, 1+\langle Q,Q\rangle)}{1-|Q|^2} \right\rangle$$

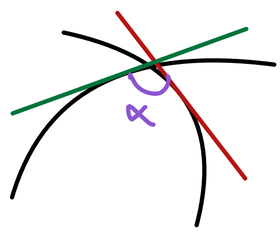
$$= \frac{1}{(1-|P|^2)(1-|Q|^2)} \cdot \big(-4\langle P,Q\rangle + 1 + \langle P,P\rangle + \langle Q,Q\rangle + \langle P,P\rangle\langle Q,Q\rangle\big)$$

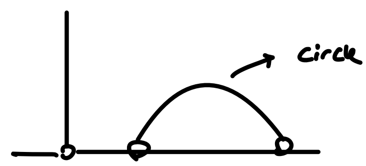
$$= (1-\langle P,P\rangle)(1-\langle Q,Q\rangle) + 2\langle P-Q,\, P-Q\rangle = 1 - \langle P,P\rangle - \langle Q,Q\rangle + \langle P,P\rangle\langle Q,Q\rangle$$

We'll continue from Anderson book

$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$





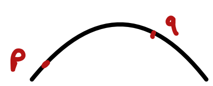


hyperbolic line $\quad \mathbb{H} \cap$
→ euclidean line $\perp$ x-axis
→ euclidean circle $\perp$ x-axis

proposition $\quad p, q \in \mathbb{H}$, $p \neq q$ there posses a unique hyperbolic line through p and q

$\operatorname{Re}(p) = \operatorname{Re}(q)$

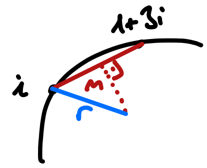
$\operatorname{Re}(p) > \operatorname{Re}(q)$



ex write down equations of the hyp. lines equations

a) $-1+2i$, $-1+7i$ answer $x=-1$

b) i, $1+3i$ $\longrightarrow$



$m_{pq} = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$, $m_{mo} = \frac{-1}{2}$

$m = \frac{i+1+3i}{2} = \left(\frac{1}{2}, 2\right) \to y-2 = \frac{-1}{2}\left(x-\frac{1}{2}\right)$

$O = MO \cap x\text{-axis}$ $\quad -2 = -\frac{1}{2}\left(x-\frac{1}{2}\right)$ $\quad x=\frac{9}{2}$ so $O\left(\frac{9}{2}, 0\right)$

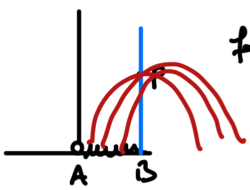
$r = \sqrt{\left(\frac{9}{2}-0\right)^2+(0-1)^2} = \frac{\sqrt{85}}{2} \Rightarrow \ell_{pq} = \left\{ z\in\mathbb{C} \mid \left|z-\frac{9}{2}\right| = \frac{\sqrt{85}}{2},\ \mathrm{Im}(z)>0 \right\}$

$\quad \|$

$d(P,O)$

Defn two hyperbolic lines ℓ, ℓ' are parallel if $\ell \cap \ell' = \emptyset$

Thm $\ell \subset \mathbb{H}$ hyperbolic line $P \in \mathbb{H}$, $P \notin \ell$ then there are infinitely many lines through P parallel to ℓ.

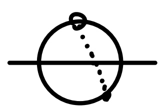


for any $Q \in [A,B]$

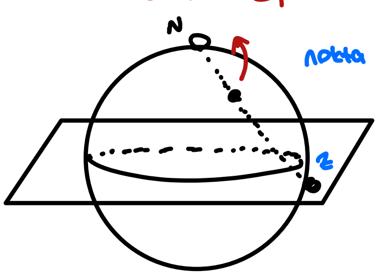
05.08.2024 (Anderson, Springer)

$\mathbb{H} = \{ z\in\mathbb{C} \mid \mathrm{Im}(z)>0 \}$ upper half plane

(1.2) Riemann Sphere $\overline{\mathbb{C}}$



1-1 onto → Ster. Proj.



nokta N ye yaklaştıkça projection $|z| \to \infty$

xy plane

$S^2 \setminus \{N\} \cong \mathbb{C}$ (1-1 onto)

$\overline{\mathbb{C}} = \mathbb{C} + \{\infty\}$ $\quad \infty$: pt at infinity (1 pt compactification)

$z \in \overline{\mathbb{C}}$, $\varepsilon > 0$ $\longrightarrow$ $z\in\mathbb{C}$ $\quad U_\varepsilon(z) = \{ w\in\mathbb{C} \mid |w-z|<\varepsilon \}$

$z \in \overline{\mathbb{C}}$, $\varepsilon > 0$ $\longrightarrow$ $z=\infty$ $\quad U_\varepsilon(z) = \{ w\in\mathbb{C} \mid |w|>\varepsilon \} \cup \{\infty\}$

open balls

a subset $X \subset \overline{\mathbb{C}}$ is open if for any $z \in X$, $\exists\, \varepsilon_x > 0$ s.t. $\mathcal{U}_{\varepsilon_x} \subset X$

$X \subset \overline{\mathbb{C}}$ is closed if $\overline{\mathbb{C}} \setminus X$ is open

Sonsuzun etrafında n-hood olmaz ?

1) $\emptyset, \overline{\mathbb{C}}$ are open !

2) if $\mathcal{U}_i \subset \overline{\mathbb{C}}$ are open then $\bigcup \mathcal{U}_i$ is open

3) if $\mathcal{U}_1 \dots \mathcal{U}_n \subset \overline{\mathbb{C}}$ are open $\bigcap_{i=1}^{n} \mathcal{U}_i$ is open in $\overline{\mathbb{C}}$

$\mathbb{H} \subset \mathbb{C}$ is open $\Rightarrow$ $\mathbb{H} \subset \overline{\mathbb{C}}$ is open (if $X \subset \mathbb{C}$ is open, then X is open in $\overline{\mathbb{C}}$)

ex S^1 (unit circle) is closed in $\overline{\mathbb{C}}$ $\quad \overline{\mathbb{C}} \setminus S^1 = \mathcal{U}_1(0) \cup \mathcal{U}_1(\infty)$ is open !

kapalı ve sınırlı (bdd) ise $\mathbb{C}$ deki her şey $\overline{\mathbb{C}}$ de de kapalı. (kapalı in $\mathbb{C}$ $\cancel{\Rightarrow}$ kapalı in $\overline{\mathbb{C}}$)

$\mathbb{C} \to \overline{\mathbb{C}}$: one pt compactification (araştır)

↳ topological space

$\{z_n\}_{n=1}$ sequence in $\overline{\mathbb{C}}$, z_n converges to $z \in \overline{\mathbb{C}}$ if $\forall \varepsilon > 0$ there is an N

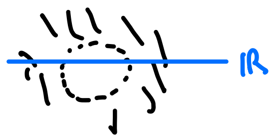
st $n \geq N \Rightarrow z_n \in U_\varepsilon(z)$ $\quad \lim z_n = z$

$\lim\left(\frac{1}{n} + i\frac{3}{n}\right) = 0$, $\lim(n + 2in) = \infty$ (+,- ayrımı yok)

$X \subset \overline{\mathbb{C}}$, $\overline{X} = \{z \in \overline{\mathbb{C}} \mid \mathcal{U}_\varepsilon(z) \cap X \neq \emptyset \ \forall \varepsilon > 0\}$ closure of X in $\overline{\mathbb{C}}$

$\overline{X} = \bigcap_{Y \supset X} Y$ (kapsayan en küçük kapalı küme) Y closed $\quad \overline{X}$ is always closed in $\overline{\mathbb{C}}$.

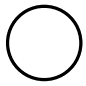
ex $\mathbb{R} \subset \overline{\mathbb{C}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$? sonsuz n-hood $\mathbb{R}$ yi kesiyor.



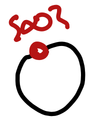
↳ is the boundary at infinity of $\mathbb{H}$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

circle in $\overline{\mathbb{C}}$ (i.e. bayutu euclidean space)

↳ either a euclidean circle in $\mathbb{C}$ 

OR

euclidean line in $\mathbb{C} \cup \{\infty\}$ 

$= \overline{L}$

L

$\overline{L}$ is the closure of L in $\overline{\mathbb{C}}$

equations of circles and lines

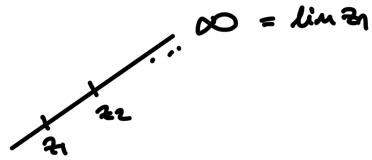
$ax + by + c = 0 \quad (a,b) \neq (0,0)$ line

$(x-x_0)^2 + (y-y_0)^2 = r^2$ circle

) write in $\mathbb{C}$ coordinates

$(z = x+iy) \ (\bar{z} = x - iy)$

$x = \dfrac{z+\bar{z}}{2} \qquad y = \dfrac{z-\bar{z}}{2}$



line : $a.\left(\dfrac{z+\bar{z}}{2}\right) + b\left(\dfrac{z-\bar{z}}{2}\right) + c = 0$

$$\underbrace{\left(\frac{a}{2} + \frac{b}{2i}\right)}_{\beta} z + \underbrace{\left(\frac{a}{2} - \frac{b}{2i}\right)}_{\bar{\beta}} \bar{z} + c = 0 \Rightarrow \beta z + \bar{\beta}\bar{z} + \gamma = 0 \quad (\beta \in \mathbb{C}, \beta \neq 0, \gamma \in \mathbb{R})$$

circle: $x^2 + y^2 - 2x_0 x - 2y_0 y + x_0^2 + y_0^2 - r^2 = 0$

$x^2 + y^2 = |z|^2 = z.\bar{z}$

$$\alpha z.\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \qquad \alpha, \gamma \in \mathbb{R}, \ \beta \in \mathbb{C}, \ \alpha \neq 0$$

$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is CONTINUOUS if for every open $\subset \overline{\mathbb{C}}$, $f^{-1}(u) \subset \overline{\mathbb{C}}$ is open

ex $j : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$z \to \frac{1}{z}$ if $z \neq 0, \infty$ is CONT proof exercise $f(z) = \frac{1}{z}$ cont.

$\left.\begin{array}{l} 0 \to \infty \\ \infty \to 0 \end{array}\right)$ MUST $\left[\frac{1}{n} \to 0 \quad \frac{1}{\frac{1}{n}} \to \infty\right]$

$j \circ j = id$ So j is its own inverse j: 1-1 onto inverstive cont j : involution

j^{-1} 1-1 onto cont ↓ homeomorphism

random polynomial $g(z) = az^n + a_1 z^{n-1} + \dots + a_n \quad n \geq 1, \ a \neq 0$

$g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$: g is CONT [for homeomorphism iff $n=1$ only]

$\infty \to \infty$

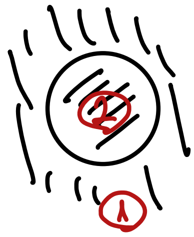
(linear polynomials $\mathbb{C} \to \mathbb{C}$ homeomorph)

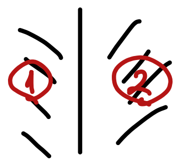
$\mathrm{Hom}(\overline{\mathbb{C}}) = \{ f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid f \text{ is homeomorphism} \}$ id, $f \circ f$ homeomorph

$\hookrightarrow$ is a group under composition. if f homeo $\to$ f^{-1} homeo

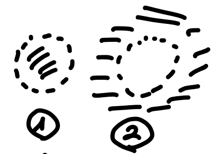
(1.3, Springer Anderson) Boundary at Infinity of $\mathbb{H}$ $(\overline{\mathbb{R}})$

A a circle in $\overline{\mathbb{C}}$: $\to$ line + $\{\infty\}$ / $\to$ circle : $\overline{\mathbb{C}} \setminus A$ has two CONNECTET component





a disk in $\overline{\mathbb{C}}$ is a connected component of the complement of a circle in $\overline{\mathbb{C}}$. ex:



conseptual no difference

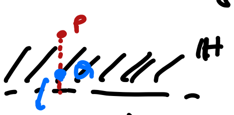
Möbius Transformations !!!

every circle in $\overline{\mathbb{C}}$ determines two disjoint disks in $\overline{\mathbb{C}}$

every disk in $\overline{\mathbb{C}}$ determines a unique circle in $\overline{\mathbb{C}}$ (its boundary)

$\mathbb{H}$: a disk in $\overline{\mathbb{C}}$ (kenarı bir doğru) whose boundary is $\overline{\mathbb{R}}$ $\to$ circle in $\overline{\mathbb{C}}$

$\overline{\mathbb{R}}$: boundary at infinity of $\mathbb{H}$.



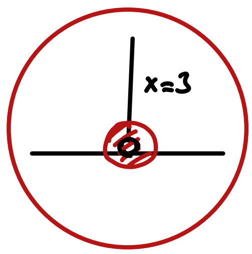
$d(P,Q) \to \infty$

poincare disk sonunda küçülüyor sonsuza gitikçe

$X \subset \mathbb{H}$ a subset $\overline{X} \cap \overline{\mathbb{R}}$: boundary at infinity of X.

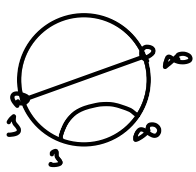
Boundary at infinity of Hyperbolic Lines

ex $\{x = 3\} \cap \mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) = 3 \quad \mathrm{Im}(z) > 0 \}$

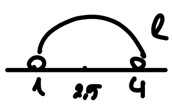


$\overline{\ell} = \ell \cup \{3, \infty\}$

$\overline{\ell} \cap \overline{\mathbb{R}} = \{3, \infty\}$



ex



$\overline{\ell} = \ell \cup \{1, 4\}$

$\overline{\ell} \cap \overline{\mathbb{R}} = \{1, 4\}$

all hyperbolic lines has two end pts at infinity

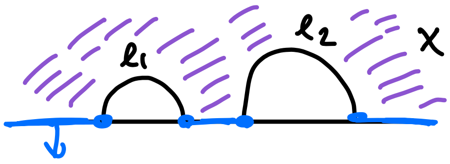
ex ℓ: $x = 1$ $(y > 0)$ has end pts at infinity $1, \infty$

ex write the eqn of the hyperbolic line with endpoints at infinity 2, 8

realize it's circle. Center $\frac{2+8}{2}$ (midpt) $= 5$ radius $= 3$

$(x-5)^2 + y^2 = 9 \quad y>0$ **OR** $|z-5| = 3$ [both answers ok]

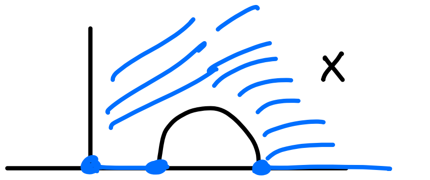
l_1, l_2 parallel



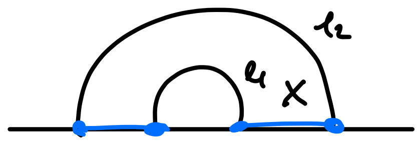
ULTRAPARALLEL

$\overline{X} \cap \overline{\mathbb{R}}$

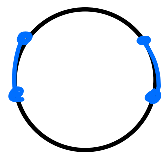
l_1 l_2 Does NOT have intersection at the pt at infinity.

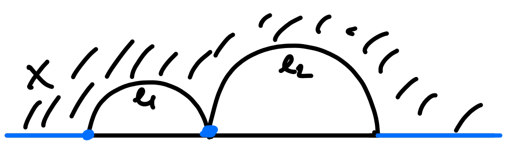


ULTRAPARALLEL $\overline{X} \cap \overline{\mathbb{R}}$

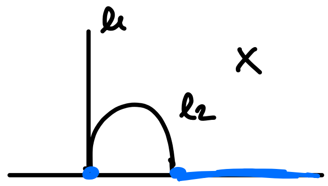


ULTRAPARALLEL





NOT ULTRAPARALLEL but parallel $\overline{X} \cap \overline{\mathbb{R}}$



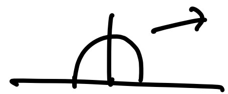
$\overline{X} \cap \overline{\mathbb{R}}$

NOT ULTRAPARALLEL but parallel

two parallel lines l_1, l_2 in $\mathbb{H}$ are called <u>ULTRAPARALLEL</u> if l_1 and l_2 have no end pts at infinity in common

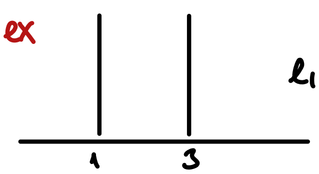
l_1, l_2 two lines in $\mathbb{H}$
- not parallel
- parallel
  - ultraparallel
  - not ultraparallel

check parallel before checking ultraparallel



→ NOT parallel (NOT ~~ultraparallel~~ ~~DEME~~)

ex

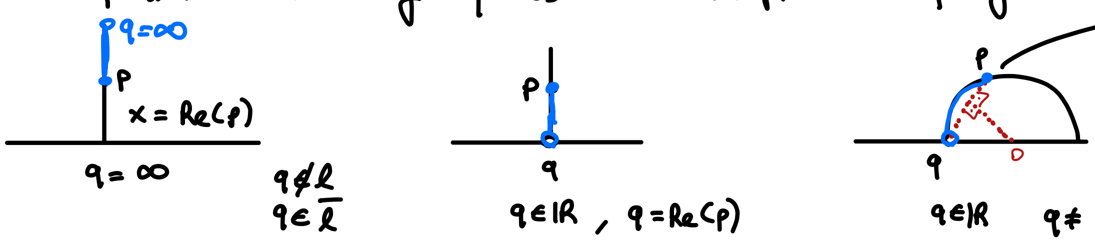


l_1 l_2 are parallel but not ultraparallel

endpts of $x=1$: $1, \infty$
end pts of $x=3$: $3, \infty$
) Sonsuzda ortak nokta var o yüzden NOT ultraparallel

Proposition $p \in \mathbb{H}$ $q \in \overline{\mathbb{R}}$ there is a unique hyperbolic line passing through p and which having q as an end pt at infinity.

Pf



hyp. ray thru p with endpt at infinity q

blue lines hyperbolic ışın

exercise Anderson 1.18 sonu

circle in $\overline{\mathbb{C}}$ → circle in $\mathbb{C}$ / line in $\mathbb{C} + \{\infty\}$

every hyp. line is contained in a unique circle in $\overline{\mathbb{C}}$

→ çemberi çembere gönderen (circle preserving)

$\text{Homeo}^c(\overline{\mathbb{C}}) = \{ f \in \text{Homeo}(\overline{\mathbb{C}}) \mid f(A) \text{ is a circle in } \overline{\mathbb{C}} \text{ if } A \text{ is a circle in } \overline{\mathbb{C}} \}$

$id \in \text{Homeo}^c(\overline{\mathbb{C}})$, $\text{Homeo}^c(\overline{\mathbb{C}})$ is closed under composition.

we do not yet know $\text{Homeo}^c(\overline{\mathbb{C}})$ is a group under composition (it is though)

$\overline{\mathbb{C}}$: küre (Riemann) *

Proposition $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$z \to az+b$ $a,b \in \mathbb{C}$, $a \neq 0$

$\infty \to \infty$

$f^{-1} \in \text{Homeo}^c(\overline{\mathbb{C}})$

$f \in \text{Homeo}^c(\mathbb{C})$

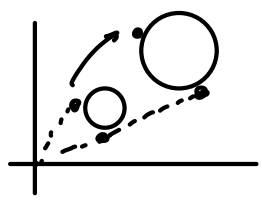
Pf $a = r.e^{i\theta}$ $r>0$ $\theta \in \mathbb{R}$

$f^{-1}(z) = \frac{z-b}{a} \Rightarrow \frac{1}{a}.z + \left(\frac{-b}{a}\right)$

invertible

$z \longrightarrow az+b$

$z \xrightarrow{\text{scaling}} r.z \xrightarrow{\text{rotation about origin by } \theta} r.e^{i\theta}.z = az \xrightarrow{\text{translation by } b} az+b$



Proposition $J(z) = \frac{1}{z}$ $\quad J \in \text{Homeo}^c(\overline{\mathbb{C}})$

Pf suppose A is a circle in $\overline{\mathbb{C}}$ (2 way)

case I A is euclidean line that passes through origin

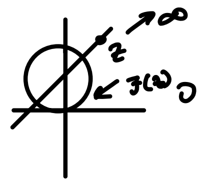
$A: \beta z + \overline{\beta}\overline{z} = 0$

$w = \frac{1}{z}$, $z = \frac{1}{w}$ $\qquad \beta \cdot \frac{1}{w} + \overline{\beta} \cdot \frac{1}{\overline{w}} = 0$

$J(A): \beta\overline{w} + \overline{\beta}w = 0$ is a euclidean line through zero $\Rightarrow$ a circle in $\overline{\mathbb{C}}$

case II $A: \beta z + \overline{\beta}\overline{z} + \gamma = 0$ $\quad \gamma \neq 0$

euclidean line not thru 0



$J(A): \beta \cdot \frac{1}{w} + \overline{\beta} \cdot \frac{1}{\overline{w}} + \gamma = 0$

$\gamma \cdot w \cdot \overline{w} + \beta\overline{w} + \overline{\beta}w = 0$ $\quad$ (euclidean circle in $\overline{\mathbb{C}}$, circle in $\overline{\mathbb{C}}$)

case III $A: \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} = 0$ $\quad \alpha \neq 0$

(euclidean circle thru 0)

$J(A): \frac{\alpha}{w\overline{w}} + \frac{\beta}{w} + \frac{\overline{\beta}}{\overline{w}} = 0$

$\overline{\beta}w + \beta\overline{w} + \alpha = 0$ (euclidean line NOT thru 0)

case IV $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$ $\quad \alpha, \gamma \neq 0$

$J(A) = \frac{\alpha}{w\overline{w}} + \frac{\beta}{w} + \frac{\overline{\beta}}{\overline{w}} + \gamma = 0$

$\gamma w\overline{w} + \overline{\beta}w + \beta\overline{w} + \alpha = 0$ (euclidean circle NOT through 0) $\square$

Möbius Transformations *

$z \to \frac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{C}$ $\quad ad - bc \neq 0$

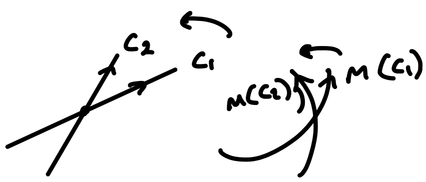
orientation preserving

Möb^+ : the set of all Möbius transformations

orientation preserving:

$M(z) = \frac{az+b}{cz+d}$

$M: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$



$M(z) = \frac{az+b}{cz+d}$, $M(\infty) = \frac{a}{c}$

[∞ if $c=0$]

$M(-\frac{d}{c}) = \infty$

Möb^+ forms a group under composition,

rewrite $\quad z \to \frac{az+b}{cz+d} = \frac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)}$, $\quad \frac{z+1}{z+2} = \frac{2z+2}{2z+4}$

$\text{id} \in \text{Möb}^+$ $\quad \text{id}(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1}$

$$M(z)=\frac{az+b}{cz+d} \qquad n(z)=\frac{pz+q}{rz+s} \qquad ad-bc\neq \qquad ps-qr\neq 0$$

$$M\circ n(z)=M(n(z))=\frac{a.(pz+q)+b(rz+s)}{c(pz+q)+d(rz+s)}=\frac{(ap+br)z+(aq+bs)}{(cp+dr)z+(cq+ds)}$$ again a Möb

$(ap+br)(cq+ds)-(aq+bs)(cp+dr)\neq 0$ Check!

$M(z)=\frac{az+b}{cz+d} \qquad M^{-1}=\frac{dz-b}{-cz+a}$ so $\Rightarrow$ Möb$^+$ is group under composition

All Möbius transformations are homeomorphism Möb$^+\subset$ Homeo $(\overline{\mathbb{C}})$

$M(z)=\frac{az+bc}{cz+d} \qquad ad-bc\neq 0$

if $c=0 \rightarrow \frac{a}{b}z+\frac{b}{d}$ $M\in \text{Homeo}^c(\overline{\mathbb{C}})$

if $c\neq 0 \rightarrow (f\circ J\circ g)(z)$ $=$ $f(z)=-(ad-bc)z+\frac{a}{c}$

$J \downarrow \frac{1}{z}$ $g(z)=c^2z+cd$

$f, J, g\in \text{Homeo}^c(\overline{\mathbb{C}})$

$\Rightarrow M\in \text{Homeo}^c(\overline{\mathbb{C}})$, Möb$^+\subset$ Homeo$^c(\overline{\mathbb{C}})$

Möb$^+$ is generated by linear polynomials and J.

Fixed pts of Möbius Transformations

$M\in$ Möb$^+$ $M\neq id$ $M(z)=z$ $M(z)=\frac{az+b}{cz+d}$

$c=0$: $\frac{a}{b}z+\frac{b}{d}=z \Rightarrow \left(\frac{a}{d}-1\right)z=\frac{-b}{a}$

$a\neq d$: 1 soln in $\mathbb{C}$ + ∞ [2 fixed points]

$a=d \Rightarrow b\neq 0 \Rightarrow 0=$something (No soln in $\mathbb{C}+\infty$) [1 fixed point]

$c\neq 0$: $\frac{az+b}{cz+d}=z$

$cz^2+(d-a)z-b=0$ [1 or 2 fixed points]

A Möbius transformations that is NOT id has ≤ 2 fixed pts. *

Transitivity given 2 triples (z_1, z_2, z_3), (w_1, w_2, w_3) of distinct pts in $\bar{\mathbb{C}}$

$\exists\, m \in \text{Möb}^+$ st $m(z_i) = w_i$ $\quad z_1 \to w_1 \quad z_2 \to w_2 \quad z_3 \to w_3$ in unique way

Möb^+ acts uniquely transitive on $\bar{\mathbb{C}}$.

$$\text{Möb}^+ \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$$
$$(m, z) \to mz$$

Special Case:

$$\underbrace{w_1}_{} \to z_1 = 0$$
$$w_2 \to z_2 = 1$$
$$\underbrace{w_3}_{\in \mathbb{C}} \to z_3 = \infty$$

$$m(z) = \frac{z - w_1}{z - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$ why it looks like cross ratio

$w_1\ w_2\ w_3 \neq \infty$

suppose one of $w_1\ w_2\ w_3$ is ∞

$w_1 = \infty$: $\quad m(z) = \dfrac{w_2 - w_3}{z - w_3} = \dfrac{0.z + (w_2 - w_3)}{1.z - w_3}$

$w_2 = \infty$: $\quad m(z) = \dfrac{z - w_1}{z - w_3}$

$w_3 = \infty$: $\quad m(z) = \dfrac{z - w_1}{w_2 - w_1}$

7 Ağustos Çarşamba

Möb Transformations $\quad m : \bar{\mathbb{C}} \to \bar{\mathbb{C}} \quad m(z) = \dfrac{az+b}{cz+d} \quad ad - bc \neq 0 \quad$ $\det = \lambda^2(ad - bc)$

iyi tanımlı değil $\quad z = \dfrac{1z+0}{0z+1}$ det = 1 $\quad z = \dfrac{2z+0}{0z+2}$ det = 4 $\quad \dfrac{az+b}{cz+d} = \dfrac{(\lambda a)z + \lambda b}{(\lambda c)z + \lambda d}$

$$\begin{array}{ccccc} & m & & n & \\ z_1 & \to & 0 & \leftarrow & w_1 \\ z_2 & \to & 1 & \leftarrow & w_2 \\ z_3 & \to & \infty & \xleftarrow{n^{-1}} & w_3 \end{array}$$

$(n^{-1} \circ m)(z_i) = z_i$

$n^{-1} \circ m = id$

$m(z) = \dfrac{z - z_1}{z - z_3} \cdot \dfrac{z_2 - z_3}{z_2 - z_1} \qquad (n^{-1} \circ m)(z_i) = w_i$

Möb^+ acts uniquely transitively on the set T of ordered triples of distinct pts of $\bar{\mathbb{C}}$

$m \in \text{Möb}^+$, A: circle in $\bar{\mathbb{C}}$, $m(A)$ is also a circle in $\bar{\mathbb{C}}$

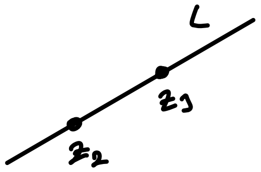
Möb^+ acts transitively on the set of circles in $\bar{\mathbb{C}}$
A, B circles in $\bar{\mathbb{C}}$. $\exists\, m \in \text{Möb}^+$ s.t. $m(A) = B$

A, B circles in $\overline{\mathbb{C}}$. $\exists\, M \in \text{Möb}^+$ s.t. $M(A) = B$
(not unique)

Pf through any three distinct pts in $\overline{\mathbb{C}}$, there posses a unique circle in $\overline{\mathbb{C}}$.

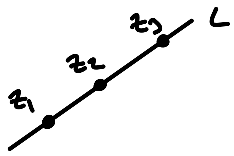
z_1, z_2, z_3

$z_1 = \infty$
$z_2, z_3 \in \mathbb{C}$

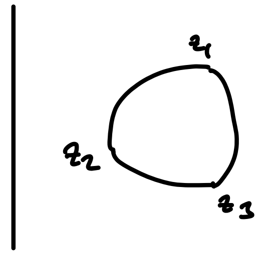


$\overline{L}$ is the unique circle in $\overline{\mathbb{C}}$ that posses through z_1, z_2, z_3

$z_1, z_2, z_3 \in \mathbb{C}$
but they lie on a euclid. line



$\overline{L}$



Pick 3 distinct pts $z_1, z_2, z_3 \in A$
" " " " $w_1, w_2, w_3 \in B$

find $M \in \text{Möb}^+$ st $M(z_i) = w_i$
$M(A) = B$

ex find $M \in \text{Möb}^+$ st $M(S^1) = \overline{\mathbb{R}}$
↳ unit circle

$0, 1, \infty \in \overline{\mathbb{R}}$

$1, i, -1 \in S^1$ (mutlak değeri 1) find $M \in \text{Möb}^+$ st $M(1) = 0$ $M(i) = 1$ $M(-1) = \infty$

$1 \to 0$ $M(z) = \dfrac{z-1}{}$

$-1 \to \infty$ $M(z) = \dfrac{}{z+1}$

$$M(z) = \frac{z-1}{z+1} \cdot \frac{i+1}{i-1}$$

Alternatively find $M \in \text{Möb}^+$ st $M(1) = 1$ $M(i) = 2$ $M(-1) = 3$

$1 \to 0 \leftarrow 1$
$i \to 1 \leftarrow 2$ OR $M(z) = \dfrac{az+b}{cz+d}$
$-1 \to \infty \leftarrow 3$

$M(1) = 1 \Rightarrow a+b = c+d$
$M(i) = 2 \Rightarrow b + ai = 2(d+ci)$
$M(-1) = 3 \Rightarrow b - a = 3(d-c)$

$a, b, c, d \in \mathbb{C}$
$b = 2d$ yazamazsın

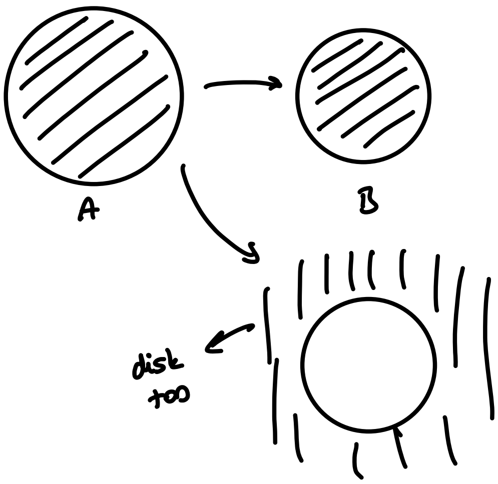
$$\begin{array}{rl} & a+b = c+d \\ + & -a+b = -3c+3d \\ \hline & 2b = -2c+4d \end{array}$$

$\boxed{b = -c+2d} \Rightarrow a = b + 3(c-d)$
$\boxed{a = 2c - d}$

now, $-c + 2d + i(2c - d) = 2d + i \cdot 2c$
$\boxed{c = -id}$

pick $d = 1 \to c = -i \to b = 2 + i \to a = -1 - 2i$

$$\Rightarrow M(z) = \frac{(-1-2i)z + (2+i)}{-iz+1}$$



$M(A) = B$ $\quad$ $M(\text{disk in } \overline{\mathbb{C}}) = \text{disk in } \overline{\mathbb{C}}$

$\overline{\mathbb{R}}$: $\quad \overline{\mathbb{C}} - \overline{\mathbb{R}} = \underbrace{\mathbb{H} \cup \mathbb{H}_-}_{\text{iki disk}}$

$J(z) = \frac{1}{z}$, $\quad J(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$

$J(0) = \infty$
$J(1) = 1$
$J(\infty) = 0$

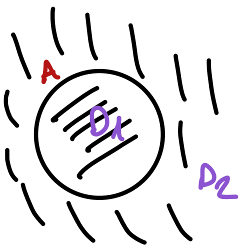
$J(\mathbb{H}) = \text{either } \mathbb{H} \text{ or } \mathbb{H}_-$

$i \in \mathbb{H}, \; J(i) = \frac{1}{i} = -i \in \mathbb{H}_-$

so $J(\mathbb{H}) = \mathbb{H}_-$

$J(\mathbb{H}_-) = \mathbb{H}$

yukarı aşağıya , aşağı yukarıya

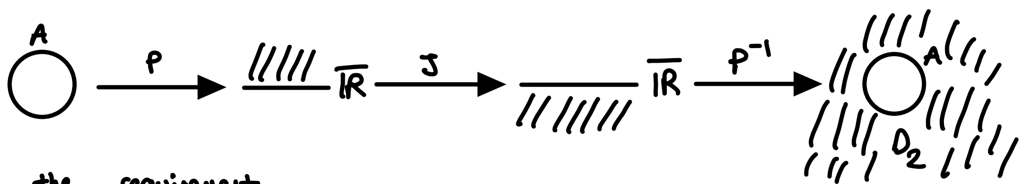


we want $M \in \text{Möb}^+$ st

$M(A) = A \qquad M(D_1) = D_2$

$M(D_2) = D_1$

find $p \in \text{Möb}^+$ st $p(A) = \overline{\mathbb{R}}$



$M = p^{-1} \circ J \circ p \in \text{Möb}^+$ satisfies the requirement

so Möb^+ acts transitively on the set of disks in $\overline{\mathbb{C}}$.

Cross - Ratio

$z_1, z_2, z_3, z_4 \in \mathbb{C}$ distinct $\quad [z_1, z_2 ; z_3, z_4] = \dfrac{(z_1 - z_4)\cdot(z_3 - z_2)}{(z_1 - z_2)\cdot(z_3 - z_4)}$!

cross ratio $\neq 0, \infty$

$[\infty, z_2 ; z_3, z_4] = \dfrac{z_3 - z_2}{z_3 - z_4}$ $\qquad$ $[z_1, z_2 ; \infty, z_4] = \dfrac{z_1 - z_4}{z_1 - z_2}$

$[z_1, \infty ; z_3 - z_4] = \dfrac{z_1 - z_4}{z_3 - z_4}$ $\qquad$ $[z_1, z_2 ; z_3, \infty] = \dfrac{z_3 - z_2}{z_1 - z_2}$

the cross ratio is defined for 4 distinct pts in $\overline{\mathbb{C}}$.

! the cross-ratio depends on the ordering of z_1, z_2, z_3, z_4

proposition $M \in Möb^+$, $[z_1,z_2;z_3,z_4] = [M(z_1),M(z_2);M(z_3),M(z_4)]$ preserving Möb

pf $Möb^+$ is gen by $az+b$, J

$\underline{az+b}$:

$$[az_1+b,\, az_2+b\,;\, az_3+b,\, az_4+b] = \frac{(az_1+b-(az_4+b))\cdot(az_3+b-(az_2+b))}{(az_1+b-(az_2+b))\cdot(az_3+b-(az_4+b))} =$$

$$= \frac{a^2(z_1-z_4)(z_3-z_2)}{a^2(z_1-z_2)(z_3-z_4)} = [z_1,z_2;z_3,z_4]$$

$\underline{J}$:

$$[J(z_1),J(z_2)\,;\,J(z_3),J(z_4)] = \frac{\left(\frac{1}{z_1}-\frac{1}{z_4}\right)\cdot\left(\frac{1}{z_3}-\frac{1}{z_2}\right)}{\left(\frac{1}{z_1}-\frac{1}{z_2}\right)\cdot\left(\frac{1}{z_3}-\frac{1}{z_4}\right)} = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)} = [z_1,z_2;z_3,z_4]$$

$\square$

! $\overline{[z_1,z_2;z_3,z_4]} = [\bar z_1,\bar z_2;\bar z_3,\bar z_4]$

cross-ratio is NOT invariant under conjugation.

proposition $z_1\ z_2\ z_3\ z_4 \in \overline{\mathbb{C}}$ distinct. Then z_1,z_2,z_3,z_4 lie on a circle in $\mathbb{C}$

iff $[z_1,z_2;z_3,z_4]$ Real.

pf find $M \in Möb^+$ st $M(z_1)=0$ $M(z_2)=1$ $M(z_3)=\infty$

z_1,z_2,z_3,z_4 lie on a circle $\Leftrightarrow$ $M(z_1), M(z_2), M(z_3), M(z_4)$ lie on on a circle in $\overline{\mathbb{C}}$.

$M(z_1), M(z_2), M(z_3), M(z_4)$ lie on on a circle in $\overline{\mathbb{C}}$. $\Leftrightarrow M(z_4)\in\mathbb{R} \Leftrightarrow$

$$\Leftrightarrow [\underset{M(z_1)}{\infty},\underset{M(z_2)}{0};\underset{M(z_3)}{1},M(z_4)] = \frac{1}{1-M(z_4)} \in \mathbb{R}$$

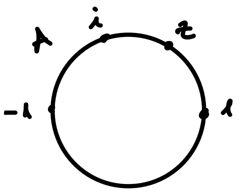
$\Leftrightarrow [z_1,z_2;z_3,z_4]\in\mathbb{R}$

exercise determine whether $2+3i$, $-2i$, $1-i$, 4 lie on a circle in $\overline{\mathbb{C}}$.

$$[2+3i,-2i;1-i,4] = \frac{(-2+3i)(1+i)}{(2+5i)(-3-i)} = \frac{-5+i}{-1-17i} \notin \mathbb{R}$$

No they do NOT lie on circle

ex determine the eqn of circle A in $\overline{\mathbb{C}}$ through $1, i, -1$



$$z \in A \iff [1, i ; -1, z] \in \mathbb{R} \iff \frac{(1-z)(-1-i)}{(1-i)\cdot(-1-z)} \in \mathbb{R}$$

$$\iff \frac{(1-z)(-1-i)}{(1-i)(-1-z)} = \frac{(1-\bar{z})(-1-i)}{(1-i)(-1-\bar{z})}$$

$$\iff (z-1)(\bar{z}+1)(\cancel{-2i}) = (z+1)(\bar{z}-1)\cdot(\cancel{2i})$$

$$\iff -(z\bar{z} + z - \bar{z} - 1) = z\bar{z} - z + \bar{z} - 1$$

$$\iff 2z\bar{z} = 2 \iff z\bar{z} = 1$$

$$\downarrow$$

$$x^2 + y^2 = 1 \quad \text{unit circle}$$

Matrix Form of Möbius Transformations

$$M(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C} \qquad ad - bc \neq 0$$

$$N(z) = \frac{pz+q}{rz+s}$$

$$\hookrightarrow (M \circ N)(z) = \frac{a \cdot \frac{pz+q}{rz+s} + b}{c \cdot \frac{pz+q}{rz+s} + d} = \frac{(ap+br)z + (aq+bs)}{(cp+dr)z + (cq+ds)}$$

look at as matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

$$GL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det A \neq 0\} \text{ group under multiplication } \xrightarrow{\Phi} \text{Möb}^+$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\Phi]{\hspace{6em}} \frac{az+b}{cz+d}$$

Φ is a surjective group homomorphism

$$\ker(\Phi) = \left\{A \in M_{2\times 2}(\mathbb{C}) \mid \det A \neq 0, \frac{az+b}{cz+d} = z \ \ \forall z \in \overline{\mathbb{C}}\right\}$$

$$\ker(\Phi) \cong \mathbb{C}^* \to GL_2(\mathbb{C})$$

$$(aI)(bI) = (ab)I$$

suppose $\frac{az+b}{cz+d} = z \quad \forall z \in \overline{\mathbb{C}}$

$$\left.\begin{array}{lll} z = \infty & \frac{a}{c} = \infty & \Rightarrow c = 0 \\ z = 0 & \frac{b}{d} = 0 & \Rightarrow b = 0 \\ z = 1 & \frac{a}{d} = 1 & \Rightarrow a = d \end{array}\right\} A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \cdot I_{2\times 2}$$

$Möb^+ \cong GL_2(\mathbb{C})/\mathbb{C}^*$

$Möb^+ \cong PGL_2(\mathbb{C})$

$PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/\mathbb{C}^*$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C})$ $\quad A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$M(z) = \frac{az+b}{cz+d}$, $M^{-1}(z) = \frac{dz-b}{-cz+a}$ $\quad \det(M) = ad-bc$

NOT well Defined

$\frac{az+b}{cz+d} = \frac{(\lambda a)z+\lambda b}{(\lambda c)z+\lambda d} \Rightarrow \det = \lambda^2(ad-bc)$

Every Möb transformation can be well be written – in two ways – so that $\det = 1$
we call that normalization

ex Normalize $M(z) = \frac{z+4}{-z+9}$ $\quad \det = 13 \rightarrow$ $\underbrace{M(z) = \frac{\frac{1}{\sqrt{13}}z + \frac{4}{\sqrt{13}}}{-\frac{1}{\sqrt{13}}z + \frac{9}{\sqrt{13}}}}_{\det = 1}$

OR
$\underbrace{M(z) = \frac{\frac{-1}{\sqrt{13}}z - \frac{4}{\sqrt{13}}}{\frac{1}{\sqrt{13}}z - \frac{9}{\sqrt{13}}}}_{\det = 1}$

Trace of Möb Matrix

$tr\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d$ $\quad T: Möb^+ \longrightarrow \mathbb{C}$

$M(z) = \frac{az+b}{cz+d} \longrightarrow (a+d)^2$ $\quad ad-bc = 1$

! $tr(A.B) = tr(B.A)$

if $\det(A) \neq 0 \Rightarrow tr(A^{-1}BA) = tr(B)$, $A^{-1}BA$ is conjugate to B.

trace is invariant under conjugation

$M, n \in Möb^+$, $T(M \circ n) = T(n \circ M)$

$T(M^{-1} \circ M \circ n) = T(n)$