

FINAL TOPICS

Recall we consider $u_t - k^2 u_{xx} = 0 \quad 0 < x < l \quad t > 0$
 $u(x, 0) = f(x) \quad 0 \leq x \leq l$
 $u(0, t) = 0 \quad u(l, t) = 0 \quad t > 0$

we have shown that for $u = X(x)T(t)$

if X satisfies $X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0$
 $T' + \lambda k^2 T = 0$

then u satisfies the eqn and boundary cond ($u(0, t) = 0, u(l, t) = 0$)

the boundary value problem (eigenvalue problem)

$$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0$$

has non trivial solns only for

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad X_n = \sin\left(\frac{n\pi}{l}x\right) \quad n = 1, 2, 3, \dots$$

Now let us find T_n

$$T_n' + \lambda_n k^2 T_n = 0 \quad \int \frac{dT_n}{T_n} = \int -\lambda_n k^2 dt$$

$$\ln|T_n| = -\lambda_n k^2 t + C$$

$$T_n = C e^{-\lambda_n k^2 t}$$

we have set of solns $u_n = X_n T_n = c_n e^{-\left(\frac{n\pi}{l}\right)^2 k^2 t} \cdot \sin\left(\frac{n\pi}{l}x\right)$

$u = \sum_{n=1}^{\infty} u_n$ is a soln if we assume that the series convergent

and can be differentiated term by term (as many terms as we need)

later we will discuss what should $f(x)$ be to make it convergent

u satisfies boundary cond but we need also for u satisfy initial cond

$$u(x,0) = f(x) \Leftrightarrow \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{l}\right)^2 k^2 t} \sin\left(\frac{n\pi}{l} x\right) \Big|_{t=0} = f(x)$$

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l} x\right) = f(x)$$

we have two questions

1) What f s can be represented as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l} x\right) \quad \text{on } [0, l] \quad ?$$

2) if f has such a representation, how to find coefficients?

FOURIER SERIES

consider a linear space $\mathcal{H}$ of piece-wise cont fns on $[-l, l]$

on space $\mathcal{H}$, introduce an inner product

(book say complex value)
but we'll stay in $\mathbb{R}$

$$\langle f, g \rangle = \int_{-l}^l f(x) g(x) dx$$

$(\mathcal{H}, \langle, \rangle)$ is a pre Hilbert space (not complete - like $\mathcal{Q}$) [u need Lebesgue $\int$ to make it complete]

we can define a norm on $\mathcal{H}$

$$f \in \mathcal{H} \quad \|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\int_{-l}^l |f|^2 dx \right)^{1/2} : f_2 \text{ norm}$$

lemma fns $\left\{ 1, \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right) \right\}_{n=1}^{\infty}$
form an orthogonal system in $\mathcal{H}$.

recall $u \perp v \Leftrightarrow \langle u, v \rangle = 0$

a seq. $\{v_n\}$ is orthogonal seq
if $\forall n \neq m \quad v_n \perp v_m$

pf first $\cos\left(\frac{n\pi}{l}x\right) \perp \cos\left(\frac{m\pi}{l}x\right) \quad n \neq m$

$$\left\langle \cos\left(\frac{n\pi}{l}x\right), \cos\left(\frac{m\pi}{l}x\right) \right\rangle = \int_0^l \cos\left(\frac{n\pi}{l}x\right) \cos\left(\frac{m\pi}{l}x\right) dx$$

$$= \frac{1}{2} \int_{-l}^l \cos\left(\frac{n-m}{l}\pi x\right) + \cos\left(\frac{n+m}{l}\pi x\right) dx$$

$$= \frac{1}{2} \left[\frac{l}{n} \frac{1}{n-m} \sin\left(\frac{n-m}{l}\pi x\right) \Big|_{-l}^l + \frac{l}{n} \frac{1}{n+m} \sin\left(\frac{n+m}{l}\pi x\right) \Big|_{-l}^l \right] = 0$$

in similar way we can consider other cases

Thm the sequence $\left\{ 1, \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right) \right\}_{n=1}^{\infty}$

is an orthogonal basis in $\mathcal{H}$

is there
• or, **Ask**

That is for any $f \in \mathcal{H}$ we have

$$f = \frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right)$$

where the series converges to f in the norm $\|\cdot\|_2$.

$$\left\| f - \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right) \right) \right\|_2 \rightarrow 0 \quad \text{no need to prove now}$$

Remark $\|1\|_2 = \sqrt{2l}$, $\|\cos\frac{n\pi}{l}x\|_2 = \sqrt{l}$, $\|\sin\frac{n\pi}{l}x\| = \sqrt{l}$

indeed $\|1\|_2 = \left(\int_{-l}^l 1^2 dx \right)^{1/2} = \sqrt{2l}$ others exercise

how to find coeff?

let $\{v_n\}$ be an orthogonal seq and

$$v = \sum_{n=1}^{\infty} c_n v_n \quad \text{consider } \langle v, v_m \rangle = \left\langle \sum_{n=1}^{\infty} c_n v_n, v_m \right\rangle = \sum_{n=1}^{\infty} c_n \underbrace{\langle v_n, v_m \rangle}_{\substack{n \neq m \\ = 0}} = \sum_{v_n \perp v_m} c_n \langle v_n, v_m \rangle$$

$$\langle v_n, v_m \rangle = c_m \langle v_m, v_m \rangle$$

$$c_m = \frac{\langle v_n, v_m \rangle}{\langle v_m, v_m \rangle} = \frac{\langle v, v_m \rangle}{\|v_m\|^2}$$

so for our system

$$\frac{a_0}{2} = \frac{1}{2l} \int_{-l}^l 1 f(x) dx \Rightarrow a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi}{l} x\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx$$

Remark since $\left\{1, \cos\left(\frac{n\pi}{l} x\right), \sin\left(\frac{n\pi}{l} x\right)\right\}$ form a basis

$$\text{the equality } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l} x\right) + b_n \sin\left(\frac{n\pi}{l} x\right)$$

$$\text{norm of } f \text{ can be calculated as } \frac{1}{l} \|f\|^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$$

Parseval Equality

$$\|f\|^2 = \frac{a_0}{2} (2l) + \sum_{n=1}^{\infty} |a_n|^2 l + |b_n|^2 l$$

$$\text{The series } \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{l} x\right) + b_n \sin\left(\frac{n\pi}{l} x\right) \right) : \text{ Fourier Series}$$

ex find the Fourier Series of $\cos^2 x$ on $[-\pi, \pi]$ $[-1, 1]$ preferred
 $[-\pi, \pi]$

$$\cos^2 x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$1 + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \cos 2x = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad \text{is the Fourier Series} \quad a_0 = 1 \quad a_2 = \frac{1}{2}$$

all other coeff zero

24 Nisan Perseus

Recall we have a linear space $\mathcal{H}$ of piecewise cont. fns on $[-l, l]$

with inner product

$$\langle f, g \rangle = \int_{-l}^l f(x) \cdot g(x) dx$$

The space $\mathcal{H}$ has a orthogonal basis $\{1, \cos(\frac{\pi}{l}x), \sin(\frac{\pi}{l}x)\}_{n=1}^{\infty}$

$\forall f \in \mathcal{H}$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx \quad a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

$n = 1, 2, 3, \dots$

the series converges to f in $\|\cdot\|_2$ norm $\left(\|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\int_{-l}^l f^2(x) dx \right)^{1/2} \right)$

$$\text{moreover, } \|f\|_2^2 = \left(\frac{a_0}{2}\right)^2 2l + \sum_{n=1}^{\infty} |a_n|^2 l + |b_n|^2 l$$

(Note: $\neq$ it's + or .)

$$\|1\|_2^2 = 2l$$

$$\left\| \cos\left(\frac{n\pi}{l}x\right) \right\|_2^2 = l$$

$$\left\| \sin\left(\frac{n\pi}{l}x\right) \right\|_2^2 = l$$

note $x = \sum c_n u_n$

$$\langle x, x \rangle = \left\langle \sum c_n u_n, \sum c_n u_n \right\rangle$$

$$\|x\|_2^2 = \sum c_n^2 \|u_n\|_2^2$$

$$\langle u_m, u_n \rangle = 0$$

$m \neq n$

ex a) find the Fourier Series of $f(x) = x$ on $[-\pi, \pi]$

b) show that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0$$

odd

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \underbrace{\cos(nx)}_{\text{odd}} \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \underbrace{\sin(nx)}_{\substack{\text{even} \\ (\text{odd} \cdot \text{odd})}} \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) \, dx = \frac{2}{\pi} \left(-\frac{1}{n} x \cos(nx) \right)_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx) \, dx$$

$$u = x \quad du = 1$$

$$v' = \sin(nx) \, dx \quad v = -\frac{1}{n} \cos(nx)$$

$$= - \left(\frac{2\pi}{\pi n} \cos(\pi n) \right) + \frac{1}{n^2} \sin(nx) \Big|_0^{\pi} = -\frac{2}{n} \underbrace{\cos(\pi n)}_{(-1)^n} = (-1)^{n+1} \frac{2}{n}$$

so, $x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)$

b) use Parseval Equality

$$\|x\|_2^2 = \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{2}{n} \right|^2 \pi = \sum_{n=1}^{\infty} \frac{4\pi}{n^2}$$

$$\|x\|_2^2 = \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{2}{3} \pi^3$$

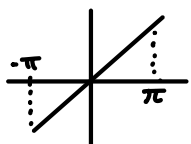
if $\frac{2}{3} \pi^3 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$ then $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Pointwise Convergence of Fourier Series

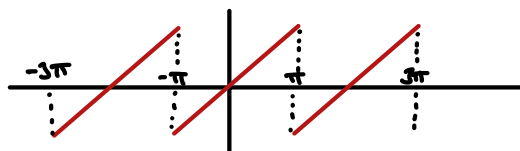
we have defined on $[-l, l]$ it is convenient to extend f to $(-\infty, \infty)$ as

$2l$ -periodic function

ex $f(x) = x$ on $[-\pi, \pi]$



the extension of f
 $f(x) = f(x + 2\pi n) \quad n \in \mathbb{Z}$



Thm let f and f' be $2l$ -periodic piecewise cont fns on $\mathbb{R}$

let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)$ be the Fourier Series of f .

if f is cont at x then $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) = f(x)$ the series converges to the value of f at x .

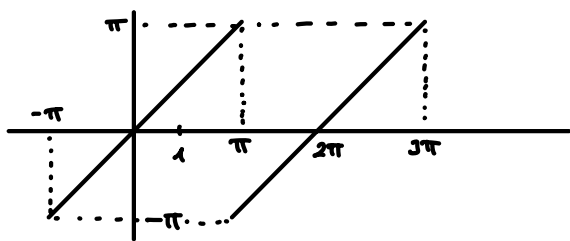
if f is discont at x then $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) = \frac{f(x^+) + f(x^-)}{2}$

where $\lim_{\varepsilon \rightarrow 0^+} f(x+\varepsilon) = f(x^+)$ and $\lim_{\varepsilon \rightarrow 0^+} f(x-\varepsilon) = f(x^-)$

ex we know $f = x$ on $[-\pi, \pi]$ then f has the Fourier Series

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx), \text{ find values } F(0), F(1), F(\pi), F(37\pi + \frac{\pi}{2})$$

$f = x$, $f' = 1$ on $[-\pi, \pi]$ piecewise cont, we can apply the prev thm.



$$x=0 \quad f \text{ cont} \quad \underline{F(0)} = f(0) = 0$$

$$x=1 \quad f \text{ cont} \quad \underline{F(1)} = f(1) = 1$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(n) = 1$$

At $x = \pi$ we consider the $2l = 2\pi$ periodic extension of f

$$\underline{F(\pi)} = \frac{f(\pi^+) + f(\pi^-)}{2} = 0$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \underbrace{\sin(n\pi)}_0 = 0$$

$$\underline{F(37\pi + \frac{\pi}{2})} = F(37\pi + \frac{\pi}{2} - 19 \cdot 2\pi) = F(-\frac{\pi}{2}) =$$

$$x = -\frac{\pi}{2} \quad f \text{ is cont} \quad F(37\pi + \frac{\pi}{2}) = F(-\frac{\pi}{2}) = f(-\frac{\pi}{2})$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(-\frac{n\pi}{2}) = -\frac{\pi}{2}$$

$$\sin(-\frac{n\pi}{2}) = (-1)^n$$

$$\sin\left(\frac{n\pi}{2}\right) =$$

$$n = 2k \quad \sin\left(\frac{2k\pi}{2}\right) = 0$$

$$n = 2k+1 \quad \sin\left(\frac{(2k+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \pi k\right) = (-1)^k$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n} \sin\left(\frac{n\pi}{2}\right) = \sum_{k=1}^{\infty} (-1)^{2k+1} \frac{2}{2k+1} \sin\left(\frac{(2k+1)\pi}{2}\right) = -\frac{\pi}{2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

ex let $f(x) = \begin{cases} 1 & [-4, -2] \\ x^2 & (-2, 2) \\ x & [2, 4] \end{cases}$ on $[-4, 4]$. let $F(x)$ be the Fourier Series of f .

find $F(10), F(-18), F(67)$

we have $2L = 8$ periodic fn.

$$F(10) = F(10-8) = f(2)$$

$$\begin{aligned} f \text{ discontinuous at } x=2 \quad & \begin{aligned} f(2^+) &= x|_2 = 2 \\ f(2^-) &= x^2|_2 = 4 \end{aligned} \quad \left. \vphantom{\begin{aligned} f(2^+) &= x|_2 = 2 \\ f(2^-) &= x^2|_2 = 4 \end{aligned}} \right\} F(10) = F(2) = \frac{f(2^+) + f(2^-)}{2} = 3 \end{aligned}$$

$$F(-18) = F(-18 + 2 \cdot 8) = F(-2)$$

$$\begin{aligned} f \text{ discontinuous at } x=-2 \quad & \begin{aligned} f(-2^+) &= x^2|_{-2} = 4 \\ f(-2^-) &= 1|_{-2} = 1 \end{aligned} \quad \left. \vphantom{\begin{aligned} f(-2^+) &= x^2|_{-2} = 4 \\ f(-2^-) &= 1|_{-2} = 1 \end{aligned}} \right\} F(-18) = F(-2) = \frac{4+1}{2} = 2.5 \end{aligned}$$

$$F(67) = F(67 - 8 \cdot 8) = F(3)$$

$$f \text{ continuous at } x=3 \quad F(67) = F(3) = f(3) = x|_3 = 3$$

$$\begin{aligned} F(4) \text{ discontinuous at } x=4 \quad & \begin{aligned} f(4^-) &= x|_4 = 4 \\ f(4^+) &= f(-4^+) = 1|_{-4} = 1 \end{aligned} \quad \left. \vphantom{\begin{aligned} f(4^-) &= x|_4 = 4 \\ f(4^+) &= f(-4^+) = 1|_{-4} = 1 \end{aligned}} \right\} F(4) = \frac{f(4^+) + f(4^-)}{2} = 2.5 \end{aligned}$$

Subtract period
 $T=8$

Functional Series

Differentiation and integration of functional series

Defn a functional series $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[0, b]$ if

$$\forall \epsilon > 0 \quad \exists N \text{ s.t. } \forall k > N \quad \left| \sum_{n=k}^{\infty} g_n(x) \right| < \epsilon \quad \forall x \in [0, b]$$

Thm let $\sum_{n=1}^{\infty} g_n(x)$ where g_n are defined on $[a, b]$

if $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[a, b]$ and $g_n(x)$ cont then

$g(x) = \sum_{n=1}^{\infty} g_n(x)$ is cont on $[a, b]$

Moreover, $\int_a^b g(x) dx = \sum_{n=1}^{\infty} \int_a^b g_n(x) dx$

$$\left[\int_a^b \left(\sum_{n=1}^{\infty} g_n(x) \right) dx = \sum_{n=1}^{\infty} \left(\int_a^b g_n(x) dx \right) \right]$$

Thm consider $\sum_{n=1}^{\infty} g_n(x)$ where $g_n(x)$ defined on $[a, b]$. if $g_n(x)$ are diff on $[a, b]$ and $\sum_{n=1}^{\infty} g'_n(x)$ converges uniformly on $[a, b]$ and

$\sum_{n=1}^{\infty} g_n(x)$ converges at least at 1 pt of $[a, b]$ then $\sum_{n=1}^{\infty} g_n(x)$ defines a diff function

on $[a, b]$, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ and $g'(x) = \sum_{n=1}^{\infty} g'_n(x)$

Remark to determine uniform convergence, we can use Weierstrass M-test

Suppose $0 < |g_n(x)| < a_n \quad \forall x \in [a, b]$

if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} g_n(x)$ uniformly conv on $[a, b]$

29 Nison Solu

Recall we have f cont on $[-l, l]$. we construct Fourier Series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)$$

This series converges to f in $\|\cdot\|_2$ norm

if f and f' are piecewise cont, we have pointwise convergence

Assume f is extended to $\mathbb{R}$ as $T=2l$ periodic function $\forall x \in \mathbb{R}$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) = \begin{cases} f(x) & \text{if } f \text{ cont at } x \\ \frac{f(x^+) + f(x^-)}{2} & \text{if } f \text{ has jump discontinuity} \end{cases}$$

$$|a_n|^2 + |b_n|^2 ???$$

When can we have

$$\int_a^b f(x) dx = \int_a^b \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right) dx$$

$$\stackrel{?}{=} \int_a^b \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \int_a^b \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right] dx$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right] \stackrel{?}{=} \sum_{n=1}^{\infty} \frac{d}{dx} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right]$$

$$\sum_{n=1}^{\infty} -a_n \frac{n\pi}{l} \sin\left(\frac{n\pi}{l}x\right) + b_n \frac{n\pi}{l} \cos\left(\frac{n\pi}{l}x\right)$$

Not always!

we need estimation on Fourier coefficients

Lemma let $f(x)$ be $T=2l$ periodic differentiable function (if we consider f on $[-l, l]$ we have $f(-l) = f(l)$) and $f'(x)$ be piecewise cont.

then the series $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges

Pf $f'(x)$ is piecewise cont, it has a Fourier Series

$$f'(x) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \left(\tilde{a}_n \cos\left(\frac{n\pi}{l}x\right) + \tilde{b}_n \sin\left(\frac{n\pi}{l}x\right) \right)$$

we know $\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} (|\tilde{a}_n|^2 + |\tilde{b}_n|^2)$ from Parseval equality

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\text{and } a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx$$

$$\begin{aligned} u &= f(x) & dv &= \cos\left(\frac{n\pi}{l}x\right) dx \\ du &= f'(x) dx & v &= \frac{l}{n\pi} \sin\left(\frac{n\pi}{l}x\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow a_n &= f(x) \frac{1}{n\pi} \sin\left(\frac{n\pi}{l}x\right) \Big|_{-l}^l - \frac{1}{n\pi} \int_{-l}^l f'(x) \sin\left(\frac{n\pi}{l}x\right) dx \\ &= \underbrace{-\frac{1}{n\pi} \int_{-l}^l f'(x) \sin\left(\frac{n\pi}{l}x\right) dx}_{l \cdot \tilde{b}_n} = -\frac{l}{n\pi} \cdot \tilde{b}_n \end{aligned}$$

$$|a_n| \leq \left| \frac{l}{n\pi} \tilde{b}_n \right| \leq \left(\frac{l}{\pi}\right)^2 \cdot \frac{1}{n^2} + \left(\frac{l}{\pi}\right)^2 |\tilde{b}_n|^2 \quad \left(xy \leq \frac{x^2+y^2}{2} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} |\tilde{b}_n|^2 \text{ convergent by comparison test}$$

$$\sum a_n \text{ converges}$$

$$\text{in the same way we can prove } \sum_{n=1}^{\infty} |b_n| \text{ converges.}$$

$$\text{Hence } \sum_{n=1}^{\infty} |a_n| + |b_n| \text{ converges } \square. \text{ We can generalize the prev lemma.}$$

Thm Let $f(x)$ be $T=2l$ periodic fn. (if consider f on $[-l, l] \Rightarrow f(-l) = f(l)$)

suppose $f, f', \dots, f^{(k)}$ are cont and $f^{(k+1)}$ is piecewise cont. Then for the

Fourier Series of f , $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)$ we have $\sum_{n=1}^{\infty} n^k (|a_n| + |b_n|)$ converges

ex let f be $T=2\pi$ periodic fn. st f', f'', f''' are cont. let f has the Fourier Series $\sum_{n=1}^{\infty} b_n \sin(nx)$. Show that $f''(x) = \sum_{n=1}^{\infty} -n^2 b_n \sin(nx) = \left(\sum_{n=1}^{\infty} \frac{d^2}{dx^2} (b_n \sin(nx)) \right)$

① f and f' cont, the series $\sum_{n=1}^{\infty} |b_n|$ converges

we have $|b_n \sin(nx)| \leq |b_n|$, hence $\sum_{n=1}^{\infty} b_n \sin(nx)$ uniformly convergent on $\mathbb{R}$

② f, f', f'' cont so, $\sum_{n=1}^{\infty} n |b_n|$ converges

$$\sum_{n=1}^{\infty} \frac{d}{dx} (b_n \sin(nx)) = \sum_{n=1}^{\infty} n \cdot b_n \cos(nx) \quad \left[|n b_n \cos(nx)| \leq n |b_n| \right]$$

so, $\sum_{n=1}^{\infty} \frac{d}{dx} b_n \sin(nx)$ uniformly conv. on $\mathbb{R}$. from ① and ② by the theorem about termwise diff we have

$$f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} b_n \cdot n \cdot \cos(nx)$$

③ we know f, f', f'', f''' are cont. so $\sum_{n=1}^{\infty} n^2 |b_n|$ converges

Now take $f'(x) = \sum_{n=1}^{\infty} n b_n \sin(nx)$ (AST)

consider $\sum_{n=1}^{\infty} \frac{d}{dx} n \cdot b_n \cdot \cos(nx) = \sum_{n=1}^{\infty} -n^2 b_n \sin(nx)$ since $|-n^2 b_n \sin(nx)| < n^2 |b_n|$

we have $\sum_{n=1}^{\infty} -n^2 b_n \sin(nx)$ converges uniformly on $\mathbb{R}$ by the thm about termwise diff of a functional series we have

$$\frac{d}{dx} f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} n b_n \cos(nx) = \sum_{n=1}^{\infty} \frac{d}{dx} n b_n \cos(nx) = \sum_{n=1}^{\infty} -n^2 b_n \sin(nx)$$

Fourier Sine and Cosine Series

(I) suppose f is piecewise cont on $[-l, l]$ and odd, then

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = 0$$

$$a_1 = \frac{1}{l} \int_{-l}^l \underbrace{f(x) \cos\left(\frac{\pi}{l}x\right)}_{\text{odd}} dx = 0$$

$$a_2 = \frac{1}{l} \int_{-l}^l \underbrace{f(x) \sin\left(\frac{\pi}{l}x\right)}_{\text{even}} dx = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{\pi}{l}x\right) dx$$

Sine Series $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$

(II) suppose f is piecewise cont on $[-l, l]$ and even

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = \frac{2}{l} \int_0^l f(x) dx$$

$$a_1 = \frac{1}{l} \int_{-l}^l \underbrace{f(x) \cos\left(\frac{\pi}{l}x\right)}_{\text{even}} dx = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{\pi}{l}x\right) dx$$

$$a_2 = \frac{1}{l} \int_{-l}^l \underbrace{f(x) \sin\left(\frac{\pi}{l}x\right)}_{\text{odd}} dx = 0$$

Cosine Series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$

let us consider that $f(x)$ on $[0, l]$. we have freedom on how to extend f on $[-l, l]$

I if we extend f as odd f_{odd}

$$f_{\text{odd}} = \begin{cases} f(x) & 0 < x \leq l \\ 0 & x = 0 \\ -f(-x) & -l \leq x < 0 \end{cases}$$

Here on $[0, l]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

then f_{odd} has sine series $f = f_{\text{odd}}$ on $(0, l)$

$\Rightarrow$ we can expand $f(x)$ into sine series

II if we extend f to $[-l, l]$ as even fn

$$f_{\text{even}} = \begin{cases} f(x) & 0 \leq x \leq l \\ f(-x) & -l \leq x \leq 0 \end{cases}$$

then f_{even} has cosine series $f = f_{\text{even}}$ on $[0, l]$

Hence, on $[0, l]$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx \quad \text{and} \quad a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx$$

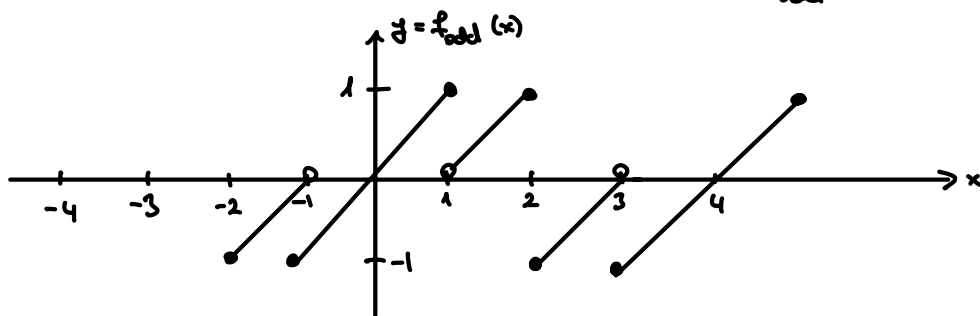
ex consider

$$f(x) = \begin{cases} x & [0, 1] \\ x-1 & x \in (1, 2] \end{cases}$$

pointwise sum
let $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right)$ and $F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right)$
find $F(1)$, $F(2)$, $F(-3)$, $F(3)$

we have thm for pointwise convergence for the Fourier Series

sine series of f is the Fourier Series of f_{odd} . We extend f_{odd} as $T=4$ period fn



$$F(1) = \frac{f_{\text{odd}}(1^+) + f_{\text{odd}}(1^-)}{2}$$

$$F(1) = \frac{1+0}{2} = \frac{1}{2}$$

$$F(2) = \frac{f_{\text{odd}}(2^+) + f_{\text{odd}}(2^-)}{2} = \frac{1+(-1)}{2} = 0$$

$$F(-3) = F(-3+4) = F(1) = \frac{1}{2} \quad / \quad F(3) = F(3-4) = F(-1) = \frac{f_{\text{odd}}(-1^-) + f_{\text{odd}}(-1^+)}{2} = \frac{0+(-1)}{2} = -\frac{1}{2}$$

6 Mays Solu

Heat Eqn $u_t - k^2 u_{xx} = 0 \quad 0 < x < l$

$$u(x, 0) = f(x) \quad 0 \leq x \leq l$$

$$u(0, t) = 0, \quad u(l, t) = 0 \quad t \geq 0$$

we look for $u = X \cdot T$ then

$$XT' - k^2 X''T = 0 \Rightarrow \frac{X''}{X} = \frac{T'}{k^2 T} = \lambda \quad \text{Separated eqn becomes}$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + k^2 \lambda T = 0$$

$$u(0, t) = 0 \Rightarrow X(0) \underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X(0) = 0$$

$$u(l, t) = 0 \Rightarrow X(l) \underbrace{T(t)}_{\neq 0} = 0 \Rightarrow X(l) = 0$$

Eigenvalue Problem

$$X'' + \lambda X = 0 \quad \begin{aligned} X(0) &= 0 \\ X(l) &= 0 \end{aligned}$$

we have non trivial soln for $\lambda_n = \left(\frac{n\pi}{l}\right)^2$

$$X_n = \sin\left(\frac{n\pi}{l}x\right) \quad n=1, 2, \dots$$

now find corresp. T_n

$$T_n' + k^2 \lambda_n T_n = 0 \Rightarrow T_n = c_n e^{-k^2 \lambda_n t}$$

$$T_n = e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \quad \text{take } c=1$$

we have particular soln $u_n(x, t)$

$$u_n(x, t) = X_n T_n = e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \cdot \sin\left(\frac{n\pi}{l}x\right)$$

since we have linear eqn

$$u(x, t) = \sum c_n u_n(x, t) \quad \text{is also a soln (formally)}$$

Now find c_n to satisfy initial cond, $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n \cdot u_n(x,0) = \sum_{n=1}^{\infty} c_n \underbrace{e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t}}_1 \cdot \sin\left(\frac{n\pi}{l}x\right) \Big|_{t=0}$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) \quad 0 \leq x \leq l$$

we know

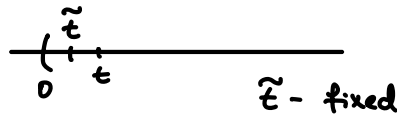
$$c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right) \quad \text{where } c_n \text{ given by above formula formally solves problem}$$

we say that we derived a formal soln

let check if u is a classical soln

first we consider $t > 0$



we take $[\tilde{t}, \infty)$ $\tilde{t} > 0$

$$|u_n(x,t)| \leq e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}}$$

Ask

let $f(x)$ be piecewise cont then its sine series coefficients are bounded

$$|c_n| < M \quad n=1,2,\dots \quad \text{then} \quad |c_n u_n(x,t)| < M \cdot e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}}$$

$$\sum_{n=1}^{\infty} M e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}} \quad \text{convergent number series}$$

$$\text{So } u(x,t) = \sum_{n=1}^{\infty} c_n \cdot u_n(x,t) \quad \text{uniformly converges on } [0,l] \times [\tilde{t}, \infty)$$

$$\frac{\partial}{\partial t} u_n(x,t) = -k^2 \left(\frac{n\pi}{l}\right)^2 e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right)$$

so $\left| c_n \frac{\partial}{\partial t} u_n(x,t) \right| \leq \tilde{M} k^2 e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}}$ where $\tilde{M} = k^2 e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}}$

$$\sum_{n=1}^{\infty} \tilde{M} n^2 e^{-k^2 \left(\frac{n\pi}{l}\right)^2 \tilde{t}} \text{ converges}$$

$$\sum_{n=1}^{\infty} c_n \frac{\partial}{\partial t} u_n(x,t) \text{ uniformly converges on } [0, l] \times [\tilde{t}, \infty)$$

Hence, $\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \frac{\partial}{\partial t} u_n(x,t) \quad (x,t) \in (0, l) \times (\tilde{t}, \infty)$

in the same way

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \frac{\partial^2}{\partial x^2} u_n(x,t) \quad (x,t) \in (0, l) \times (\tilde{t}, \infty)$$

we proved $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$

solves the eqn $u_t - k^2 u_{xx} = 0$ on $0 < x < l$
 $t > \tilde{t} \quad \forall \tilde{t} > 0 \quad \text{for any } t > 0$

$u(0,t) = 0$
 $u(l,t) = 0$ is trivially satisfied

$$u(x,0) = f(x) \iff f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l} x\right)$$

from the thm for pointwise convergence of the Fourier Series we have that the above equality hold if f is cont, $f(0) = f(l)$ and f' is piecewise cont.

Thm if f is cont on $[0, l]$, $f(0) = f(l)$ and f' piecewise cont on $[0, l]$

then the initial boundary value problem

$$u_t - k^2 u_{xx} = 0, \quad 0 < x < l, \quad t > 0$$

has a solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k^2 \left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$$

$$u(x,0) = f(x) : 0 \leq x \leq l$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \geq 0$$

$$\text{where } c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx$$

ex find formal solution of $u_{tt} = c^2 u_{xx}$ $0 < x < l$ $t > 0$

$$u_x(0,t) = 0 \quad u_x(l,t) = 0$$

$$u(x,0) = f(x) \quad u_t(x,0) = g(x)$$

assume $u(x,t) = X(x)T(t)$ and subst into eqn

$$XT'' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T'' + \lambda c^2 T = 0 \end{cases} \quad \text{with IC}$$

$$u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$u_x(l,t) = X'(l)T(t) = 0 \Rightarrow X'(l) = 0$$

we have boundary value problem (eigenvalue problem)

$$X'' + \lambda X = 0 \quad X'(0) = 0 \quad X'(l) = 0$$

Char eqn is $r^2 + \lambda = 0 \Rightarrow r = \pm \sqrt{\lambda}$ form of soln depends on the sign of λ

I $\lambda < 0$ $\lambda = -\mu^2, \mu > 0$

$$r = \pm \mu$$

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$X' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x} \quad \text{and} \quad X'(0) = 0 \Rightarrow \mu c_1 - \mu c_2 = 0$$

$$\Rightarrow c_1 = c_2$$

$$\text{Also, } X'(l) = 0 \quad \mu c_1 e^{\mu l} - \mu c_2 e^{\mu l} = 0 \Rightarrow c_1 e^{\mu l} - c_2 e^{\mu l} = 0$$

$$\det \begin{pmatrix} 1 & -1 \\ e^{\mu l} & -e^{\mu l} \end{pmatrix} \neq 0 \quad \therefore c_1 - c_2 = 0 \quad \text{only trivial soln}$$

when $\det = 0$, rows are

dependent thus you have 1 eqn inst of 2

II $\lambda = 0$ $r=0$ (double root)

$$X = c_1 x + c_2 \Rightarrow X' = c_1 \quad \left. \begin{array}{l} X'(0) = 0 \\ X'(l) = 0 \end{array} \right\} c_1 = 0, \quad c_2 \text{ is free}$$

$\Rightarrow X = c_2$ is non-trivial soln

we have $\lambda_0 = 0$ is an eigenvalue and $X_0 = 1/2$ is the corresponding eigenfunction

III $\lambda > 0$ $\lambda = \mu^2$; $\mu > 0$

$$r = \mp \sqrt{-\mu^2} = \mp i\mu \quad (\alpha + i\beta)$$

$$X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$X' = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x)$$

$$\text{and } X'(0) = 0 \Rightarrow \mu c_2 = 0 \Rightarrow c_2 = 0$$

$$X'(l) = 0 \Rightarrow -\mu c_1 \sin(\mu l) + \mu c_2 \cos(\mu l) = 0$$

$$\begin{pmatrix} 0 & 1 \\ \sin(\mu l) & \cos(\mu l) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\Rightarrow c_1 \sin(\mu l) = 0 \quad \text{for non trivial soln } c_1 \neq 0$$

$$\Rightarrow \sin(\mu l) = 0 \Rightarrow \mu l = n\pi \quad n=1, 2, \dots$$

$$\mu = \frac{n\pi}{l}$$

so, $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ are eigenvalues with corresponding eigenfunctions

$$X_n = \cos\left(\frac{n\pi}{l} x\right)$$

we have eigenvalues and eigenfunctions

$$\lambda_0 = 0 \quad X_0 = \frac{1}{2}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad X_n = \cos\left(\frac{n\pi}{l} x\right)$$

Now we'll find corresponding T_n

$$(T'' + c^2 \lambda T = 0) \quad T_n'' + c^2 \lambda_n T_n = 0$$

$$\lambda_0 = 0, T_0'' = 0 \quad T_0 = b_0 t + a_0$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad T_n'' + c^2 \left(\frac{n\pi}{l}\right)^2 T_n = 0 \xrightarrow{\text{char eqn}} r^2 + c^2 \left(\frac{n\pi}{l}\right)^2 = 0$$

$$r = \pm c \frac{n\pi}{l} i$$

$$T_n = a_n \cos\left(\frac{cn\pi}{l}t\right) + b_n \sin\left(\frac{cn\pi}{l}t\right)$$

$$u_0 = X_0 T_0 = (b_0 t + a_0) \cdot \frac{1}{2}$$

$$u_n = X_n T_n = \left(a_n \cos\left(\frac{cn\pi}{l}t\right) + b_n \sin\left(\frac{cn\pi}{l}t\right)\right) \cdot \cos\left(\frac{n\pi}{l}x\right)$$

we have

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)$$

$$= (b_0 t + a_0) \frac{1}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi}{l}t\right) + b_n \sin\left(\frac{cn\pi}{l}t\right)\right) \cdot \cos\left(\frac{n\pi}{l}x\right)$$

with initial cond. $u(x,0) = f(x)$

$$u_t(x,0) = g(x)$$

$$u(x,0) = f(x) \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$$

$$\text{hence } a_0 = \frac{2}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx \quad (2)$$

$$u_t(x,0) = g(x) \Rightarrow u_t(x,t) = b_0 \cdot \frac{1}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{cn\pi}{l} \sin\left(\frac{cn\pi}{l}t\right) + b_n \frac{cn\pi}{l} \cos\left(\frac{cn\pi}{l}t\right)\right) \cdot \cos\left(\frac{n\pi}{l}x\right)$$

$$g(x) = u_t(x,0) \underset{t=0}{=} \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \cos\left(\frac{n\pi}{l}x\right)$$

$$\text{we know (1)} \quad b_0 = \frac{2}{l} \int_0^l g(x) dx, \quad b_n \cdot \frac{cn\pi}{l} = \frac{2}{l} \int_0^l g(x) \cos\left(\frac{n\pi}{l}x\right) dx$$

$$\Rightarrow b_n = \frac{2}{cn\pi} \int_0^l g(x) \cos\left(\frac{n\pi}{l}x\right) dx$$

We found the formal soln

$$u(x,t) = \left(b_0 t + a_0\right) \frac{1}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi}{l}t\right) + b_n \sin\left(\frac{cn\pi}{l}t\right)\right) \cos\left(\frac{n\pi}{l}x\right)$$

where a_0, a_n, b_0, b_n $n=1,2,\dots$ are given by (1), (2)

8 May 1988

Recall: we find formal soln

$$u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < l \quad t > 0$$

$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \quad 0 \leq x \leq l$$

$$u_x(0,t) = 0, \quad u_x(l,t) = 0 \quad t \geq 0$$

$$u(x,t) = \left(\frac{a_0}{2} + \frac{b_0}{2}t\right) + \sum_{n=1}^{\infty} \left(a_n \cos\frac{cn\pi}{l}t + b_n \sin\frac{cn\pi}{l}t\right) \cos\frac{n\pi}{l}x$$

ex find soln of $u_{tt} - 4u_{xx} = 0 \quad 0 < x < 1 \quad t > 0$

$$u(x,0) = 1 + \cos(2\pi x) \quad u_t(x,0) = \sin^2 2\pi x \cos^2 2\pi x \quad 0 \leq x \leq 1$$

$$u_x(0,t) = u_x(1,t) = 0 \quad t \geq 0$$

we know the soln

$$u(x,t) = \left(\frac{a_0}{2} + \frac{b_0}{2}t\right) + \sum_{n=1}^{\infty} \left(a_n \cos\frac{cn\pi}{l}t + b_n \sin\frac{cn\pi}{l}t\right) \cos\frac{n\pi}{l}x$$

$$a_n, b_n ? \quad u(x,0) = 1 + \cos(2\pi x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_0 = \frac{2}{1} \int_0^1 f(x) dx \quad a_n = \frac{2}{1} \int_0^1 f(x) \cos(n\pi x) dx$$

$$1 + \cos(2\pi x) = \frac{a_0}{2} + a_1 \cos \pi x + a_2 \cos 2\pi x + a_3 \cos 3\pi x + \dots$$

$$a_0 = 2 \quad a_1 = 0 \quad a_2 = 1 \quad a_3 = 0$$

$$u_t(x,0) = \sin^2(2\pi x) \cdot \cos^2(2\pi x)$$

$$u_t(x,t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \left((-2n\pi)a_n \sin(2n\pi t) + (2n\pi)b_n \cos(2n\pi t)\right) \cos(n\pi x) \Big|_{t=0}$$

$$\sin^2(2\pi x) \cos^2(2\pi x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} 2n\pi b_n \cos(n\pi x)$$

$$\sin^2(2\pi x) \cos^2(2\pi x) = \left[\frac{1}{2} \sin 4\pi x\right]^2 = \frac{1}{4} \sin^2 4\pi x = \frac{1}{4} \cdot \frac{1 - \cos 8\pi x}{2} = \frac{1}{8} - \frac{1}{8} \cos(8\pi x)$$

$$b_0 = 1/4 \quad b_8 = -1/128\pi \quad b_n = 0 \quad n \neq 0, 8$$

$$u(x,t) = \left(1 + \frac{1}{8}t\right) + \cos(4\pi t) \cos(2\pi x) - \frac{1}{128\pi} \sin(16\pi t) \cos(8\pi x)$$

($a_2 = 1, n=2$) ($b_8 = -1/128\pi, n=8$)

Non-homogenous Eqs

idea expand all fns into a series of eigenfunctions corresponding to the homogenous problem

ex find the formal soln of

$$u_{tt} - c^2 u_{xx} = F(x, t) \quad 0 < x < l \quad t > 0$$

$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x)$$

$$u(0, t) = 0 \quad u(l, t) = 0$$

eigenfunctions: we consider the homogenous problem $u_{tt} - c^2 u_{xx} = 0$

let $u = X \cdot T$ then X will satisfy (we done it before)

$$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(l) = 0 \quad \text{and we solved this problem}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad X_n = \sin\left(\frac{n\pi}{l}x\right) \quad n = 1, 2, 3, \dots \quad \text{eigenfunction}$$

expand everything in terms of eigenfunctions for each t fixed

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi}{l}x\right) \quad \text{note: } F_n(t), f_n, g_n \text{ given}$$

$u_n(t)$ we need to find

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{l}x\right)$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{l}x\right)$$

substituting into the eqn we get

$$\sum_{n=1}^{\infty} u_n''(t) \sin\left(\frac{n\pi}{l}x\right) - c^2 \sum_{n=1}^{\infty} u_n(t) \left(-\left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi}{l}x\right)\right) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

$$u(x, 0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{n\pi}{l}x\right) = f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{l}x\right)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} u_n'(0) \sin\left(\frac{n\pi}{l}x\right) = g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{l}x\right)$$

computing the coeff of the eigenfunctions we get

$$u_n'' + c^2 \left(\frac{n\pi}{l}\right)^2 u_n = F_n(t) \quad \text{equation to find } u_n(t)$$

$$u_n(0) = f_n, \quad u_n'(0) = g_n$$

we know that $u_n = u_n^{\text{homo}} + u_n^{\text{particular}}$

$$u_n^h : u_n'' + c^2 \left(\frac{n\pi}{l} \right)^2 u_n = 0$$

u_n^p : 1 particular soln use Variation of Parameters

$$u_n^h = A_n \cos \left(\frac{cn\pi}{l} x \right) + B_n \sin \left(\frac{cn\pi}{l} x \right)$$

$$u_n^p = \frac{l}{cn\pi} \int_0^t F_n(z) \sin \left(\frac{cn\pi(t-z)}{l} \right) dz$$

$$\Rightarrow u_n(t) = A_n \cos \left(\frac{cn\pi}{l} x \right) + B_n \sin \left(\frac{cn\pi}{l} x \right) + \frac{l}{cn\pi} \int_0^t F_n(z) \sin \left(\frac{cn\pi(t-z)}{l} \right) dz$$

find A_n and B_n from $u_n(0) = f_n$, $u_n'(0) = g_n$

$$u_n(0) = f_n = A_n$$

$$u_n'(t) = -A_n \frac{cn\pi}{l} \sin \left(\frac{cn\pi}{l} t \right) + B_n \frac{cn\pi}{l} \cos \left(\frac{cn\pi}{l} t \right) + \frac{l}{n\pi c} F_n(t) \sin \left(\frac{n\pi c}{l} (t-t) \right) + \frac{l}{n\pi c} \int_0^t F_n(z) \frac{l}{n\pi c} \cos \left(\frac{cn\pi}{l} (t-z) \right) dz$$

$$u_n'(0) = \frac{cn\pi}{l} B_n = g_n \Rightarrow B_n = \frac{l g_n}{cn\pi}$$

the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \left(\frac{n\pi}{l} x \right) \quad \text{where}$$

$$u_n(t) = f_n \cos \left(\frac{cn\pi}{l} t \right) + \frac{l}{cn\pi} g_n \sin \left(\frac{cn\pi}{l} t \right) + \frac{l}{cn\pi} \int_0^t F_n(z) \sin \left(\frac{cn\pi}{l} (t-z) \right) dz$$

$$\text{let's say } \omega_n = \frac{n\pi c}{l} \Rightarrow u_n'' + \omega_n^2 u_n = F_n(t)$$

$$\text{homo soln gives } A_n \cos \omega_n t + B_n \sin \omega_n t \Rightarrow A_n = f_n, B_n = \frac{g_n}{\omega_n}$$

$$u_n(t) = f_n \cos \omega_n t + \frac{g_n}{\omega_n} \sin \omega_n t + \int_0^t \frac{\sin(\omega_n(t-s))}{\omega_n} F_n(s) ds$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \left(\frac{n\pi}{l} x \right)$$

Duhamel's Principle: superposition of responses to infinitesimal impulses

for homogeneous case, integral vanishes

ex find the formal soln of

$$u_t - u_{xx} = xt \quad 0 < x < \pi \quad t > 0$$

$$u(x, 0) = \cos x \quad 0 \leq x \leq \pi$$

$$u_x(0, t) = 0, \quad u_x(\pi, t) = 0$$

eigenfunctions

$$u_t - u_{xx} = 0, \quad u_x(0, t) = 0, \quad u_x(\pi, t) = 0$$

$$u = X \cdot T, \quad \text{subst} \quad XT' - X''T = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = -\lambda \quad \Rightarrow \quad X'' + \lambda X = 0$$

$$u_x(0, t) = 0 : X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$u_x(\pi, t) = 0 : X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0$$

the eigenvalue problem

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0$$

$$\text{eigenvalues } \lambda_0 = 0, \quad \lambda_n = n^2 \Rightarrow \text{corresp. eigenfunctions : } X_0 = \frac{1}{2}, \quad X_n = \cos(nx)$$

expand everything in terms of eigenfunctions

$$u = u_0(t) \frac{1}{2} + \sum_{n=1}^{\infty} u_n(t) \cos(nx)$$

$$F(x, t) = xt = F_0(t) \frac{1}{2} + \sum_{n=1}^{\infty} F_n(t) \cos(nx) \quad \text{cosine series}$$

$$F_0(t) = \frac{2}{\ell} \int_0^{\ell} F(x, t) dx = \frac{2}{\pi} \int_0^{\pi} xt dx = \frac{2}{\pi} t \left[\frac{x^2}{2} \right]_0^{\pi} = \pi t$$

$$F_n(t) = \frac{2}{\ell} \int_0^{\ell} F(x, t) \cos\left(\frac{n\pi}{\ell} x\right) dx = \frac{2}{\pi} \int_0^{\pi} xt \cos(nx) dx = \frac{2}{\pi} t \int_0^{\pi} x \cos(nx) dx$$

IBP $\begin{matrix} p = x \\ p' = 1 \end{matrix} \quad \begin{matrix} dq = \cos nx dx \\ q = \frac{1}{n} \sin nx \end{matrix}$

substitute everything into the eqn

$$\frac{1}{2} u_0' + \sum_{n=1}^{\infty} u_n'(t) \cos(nx) - \sum_{n=1}^{\infty} u_n(t) (-n^2 \cos(nx)) = \frac{\pi t}{2} + \sum_{n=1}^{\infty} \frac{2t}{n^2 \pi} ((-1)^n - 1) \cos(nx)$$

$$u_0(0) \cdot \frac{1}{2} + u_1(0) \cos x + u_2(0) \cos(2x) + \dots = 0 \cdot \frac{1}{2} + 1 \cdot \cos x + 0 \cdot \cos 2x + 0 \cdot \cos(3x) + \dots$$

$$= \frac{2}{\pi} t \left(\frac{1}{n} x \sin(nx) \right) \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n} \sin(nx) dx$$

$$= \frac{2}{\pi} t \left(\frac{1}{n^2} \cos(nx) \Big|_0^{\pi} \right) = \frac{2}{\pi n^2} t (\cos(n\pi) - 1)$$

$$F(x, t) = xt = \pi t \cdot \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} ((-1)^n - 1) t \cdot \cos(nx)$$

$$u(x,0) = \cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\begin{aligned} a_0 &= 0 & a_1 &= 1 \\ a_2 &= 0 & a_3 &= 0 \end{aligned}$$

cosine series

eqn for $u_n(t)$

$$n=0 \quad u_0'(t) = \frac{\pi t}{2} \quad u_0(0) = 0$$

$$n=1 \quad u_1'(t) + u_1 = \frac{2}{\pi} t (-2) \quad u_1(0) = 1$$

$$n=2,3,\dots \quad u_n'(t) + n^2 u_n(t) = \frac{2t}{n^2 \pi} ((-1)^n - 1) \quad u_n(0) = 0$$

note: $n=2k$ $u_n=0$ $k=1,2,3,\dots$ next we find $u_n(t)$ and we have a soln solving the ODEs:

$$n=0 : u_0'(t) = \frac{\pi t}{2}, \quad u_0(0) = 0 \Rightarrow u_0(t) = \frac{\pi}{4} t^2$$

$$n=1 : u_1'(t) + u_1(t) = -\frac{4}{\pi} t, \quad u_1(0) = 1 \quad \text{variation of parameters}$$

$$\frac{d}{dt}(e^t u_1) = -\frac{4}{\pi} t e^t, \quad \text{now integrate } 0 \rightarrow t$$

$$e^t u_1(t) - u_1(0) = -\frac{4}{\pi} (t e^t - e^t + 1)$$

$$\Rightarrow u_1(t) = e^{-t} \left(1 - \frac{4}{\pi}\right) - \frac{4t}{\pi} + \frac{4}{\pi}$$

$n=2,3,\dots$

$$\hookrightarrow \text{even } n (2,4,6,\dots) : \quad u_n(t) = 0$$

$$u_n'(t) + n^2 u_n(t) = 0 \quad u_n(0) = 0 \Rightarrow u_n(t) = 0$$

$$\hookrightarrow \text{odd } n (1,3,5,\dots) :$$

$$u_n' + n^2 u_n = \frac{-4}{\pi n^2} t, \quad u_n(0) = 0 \quad \mu = e^{n^2 t}$$

$$u_n(t) = e^{-n^2 t} \left(\frac{-4}{\pi n^2} \right) - \frac{4t}{\pi n^4} + \frac{4}{\pi n^6}$$

$$u(x,t) = \frac{\pi t^2}{4} + \left[e^{-t} \left(1 - \frac{4}{\pi}\right) - \frac{4t}{\pi} + \frac{4}{\pi} \right] \cos x + \sum_{\text{odd}} \left[-\frac{4}{\pi n^6} e^{-n^2 t} - \frac{4}{\pi n^4} + \frac{4}{\pi n^6} \right] \cos(nx)$$

note even modes vanishes

13.13.13 Solu

Recall we considered non homogeneous eqns, non-homog boundary cond.

idea: we change the dependent variable so that the boundary cond become homogeneous

ex transform the given problem with homogeneous boundary cond

$$u_{tt} - c^2 u_{xx} = F(x,t) \quad 0 < x < l \quad t > 0$$

$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \quad 0 \leq x \leq l$$

$$u(0,t) = \mu_1(t) \quad u(l,t) = \mu_2(t) \quad t \geq 0$$

assume $u = v + Q$

$v(x,t)$ - new independent variable

$Q(x,t)$ some particular fn

we want v to satisfy homogenous boundary cond

$$v(0,t) = 0 \quad v(l,t) = 0$$

now we find Q

subst. $u = v + Q$ into the problem

$$(v+Q)_{tt} - c^2 (v+Q)_{xx} = F(x,t)$$

$$v(x,0) + Q(x,0) = f(x) \quad v_t(x,0) + Q_t(x,0) = g(x)$$

$$v(0,t) + Q(0,t) = \mu_1(t) \quad v(l,t) + Q(l,t) = \mu_2(t)$$

$$v(0,t) = 0 \Rightarrow Q(0,t) = \mu_1(t) \quad (*)$$

$$v(l,t) = 0 \Rightarrow Q(l,t) = \mu_2(t)$$

there are many fns Q satisfying $(*)$. We choose Q linear in x

$$Q = a(t)x + b(t) \quad [\text{linear fn through } (0, \mu_1) \text{ and } (l, \mu_2)]$$

$$\text{that is } Q(x,t) = \mu_1(t) + \frac{x}{l} (\mu_2(t) - \mu_1(t))$$

for each Q we have that v satisfies

$$v_{tt} - c^2 v_{xx} = F(x,t) - (Q_{tt} - Q_{xx}) = F(x,t) - \left[\mu_1''(t) + \frac{x}{l} (\mu_2''(t) - \mu_1''(t)) \right]$$

$$v(x,0) = f - Q(x,0) = f(x) - \left[\mu_1(0) + \frac{x}{l} (\mu_2(0) - \mu_1(0)) \right]$$

$$v_t(x,0) = g - Q_t = g(x) - \left[\mu_1'(0) + \frac{x}{l} (\mu_2'(0) - \mu_1'(0)) \right]$$

$$v(0,t) = 0, \quad v(l,t) = 0$$

we have a problem with hmgns boundary cond.

ex transform the given problem to a problem with homogenous boundary cond.

$$u_t - a^2 u_{xx} = t^2 e^x \quad 0 < x < \pi \quad t > 0$$

$$u(x,0) = 0 \quad 0 \leq x \leq \pi$$

$$u_x(0,t) = t, \quad u(\pi,t) = 1 \quad t > 0$$

$$\text{Set } u = v + Q$$

$$u_x(0,t) = v_x(0,t) + Q_x(0,t) = t$$

$$v_x(0,t) = 0 \Rightarrow Q_x(0,t) = t$$

$$u(\pi,t) = v(\pi,t) + Q(\pi,t) = 1$$

$$v(\pi,t) = 0 \Rightarrow Q(\pi,t) = 1$$

$$\text{assume } Q \text{ is linear in } x \quad \text{st} \quad Q = a(t)x + b(t)$$

Sometimes you may need to pick Q quadratic. Be aware

$$Q_x(0,t) = a(t) = t$$

$$Q(\pi,t) = a(t)\pi + b(t) = 1 \Rightarrow b(t) = 1 - \pi t$$

$$\Rightarrow Q(x,t) = tx + 1 - \pi t$$

$$\text{this gives exactly } Q_x(0,t) = t$$

$$Q(\pi,t) = 1$$

$$\text{Define the new unknown with hmgns boundaries} \quad v(x,t) = u(x,t) - Q(x,t)$$

$$v_x(0,t) = u_x(0,t) - Q_x(0,t) = t - t = 0 \quad \checkmark$$

$$v(\pi,t) = u(\pi,t) - Q(\pi,t) = 1 - 1 = 0 \quad \checkmark$$

$$\text{Since } u = v + Q \quad \text{plug it in original eqn} \quad u_t - a^2 u_{xx} = t^2 e^x$$

$$(v_t + Q_t) - a^2 (v_{xx} + Q_{xx}) = t^2 e^x$$

$$v_t - a^2 v_{xx} = t^2 e^x - (Q_t - Q_{xx}) \quad \text{and} \quad Q(x,t) = 1 + xt - \pi t \Rightarrow Q_t = x - \pi, \quad Q_{xx} = 0$$

$$\Rightarrow v_t - a^2 v_{xx} = t^2 e^x - (x - \pi), \quad v(x,0) = \underbrace{u(x,0)}_0 - \underbrace{Q(x,0)}_1 = -1$$

Transformed PDE

$$v_t - a^2 v_{xx} = t^2 e^x - (x - \pi)$$

$$v_x(0,t) = 0, \quad v(\pi,t) = 0$$

$$v(x,0) = -1$$

ex Simplify the given problem to a homogeneous problem with homogeneous cond (if possible)

$$u_t - 2u_{xx} = 2e^x \quad 0 < x < 1 \quad t > 0$$

$$u(x, 0) = x$$

$$u(0, t) = 4 \quad u(1, t) = 5$$

$$\text{Set } u = v + q$$

$$(v+q)_t - 2(v+q)_{xx} = 2e^x$$

$$v_t - 2v_{xx} = 2e^x - (q_t - 2q_{xx})$$

$$\text{we want } 2e^x - (q_t - 2q_{xx}) = 0$$

$$u(0, t) + q(0, t) = 4 \Rightarrow v(0, t) = 4 - q(0, t)$$

$$\text{we want } 4 - q(0, t) = 0$$

$$u(1, t) + q(1, t) = 5 \Rightarrow v(1, t) = 5 - q(1, t)$$

$$\text{we want } 5 - q(1, t) = 0$$

$$\text{think of } q_{xx} = -e^x$$

$$q_x = -e^x + c_1$$

$$q = -e^x + c_1 x + c_2 \Rightarrow q(0) = 4 : -1 + c_2 = 4 \Rightarrow c_2 = 5$$

$$q(1) = 5 : -e + c_1 \cdot 1 + c_2 = 5 \Rightarrow c_1 = e$$

$$\text{So } q = -e^x + ex + 5$$

$$\text{thus } v_t - 2v_{xx} = 0$$

$$\text{bd cond } v(0, t) = 0, \quad v(1, t) = 0$$

$$\text{initial cond } v(x, 0) = u(x, 0) - q(x, 0) = x - (ex + 5 - e^x)$$

Elliptic Equations

Define the Laplace Operator $\Delta : \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$

for $u \in C^2(\Omega)$

$$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} \dots + u_{x_n x_n} \quad \text{in } \Omega \subset \mathbb{R}^n$$

we will consider

$$\Delta u = 0, \quad x \in \Omega \quad : \text{Laplace Eqn}$$

$$\Delta u = F(x), \quad x \in \Omega \quad : \text{Poisson Eqn}$$

with boundary cond

$$u|_{\partial\Omega} = f \quad : \text{Dirichlet boundary cond}$$

$$\frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = f \quad : \text{Neumann boundary cond}$$

$\vec{n}$ is the unit outward normal to $\partial\Omega$

$\frac{\partial u}{\partial \vec{n}}$ normal derivative (derivative in the direction of $\vec{n}$) $\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}$

uniqueness and stability for Laplace (Poisson) Eqn

Thm (weak max principle) let Ω be a bounded domain in $\mathbb{R}^n$ and

$$u \in C^2(\Omega) \cap C(\bar{\Omega})$$

Suppose $\Delta u \geq 0$ in Ω then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Pf case I Suppose $\Delta u > 0$

Note since $u \in C(\bar{\Omega})$ and $\bar{\Omega}$ closed bdd set in $\mathbb{R}^n$, u has a max pt in $\bar{\Omega}$. Claim is a max pt can not belong to Ω . Assume that x^0 is a max pt and $x^0 \in \Omega$. Since x^0 is interior max pt, $u_{x_{ii}}(x^0) \leq 0 \Rightarrow \Delta u(x^0) \leq 0$

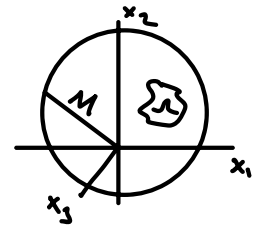
Contradiction we have $\Delta u > 0$ in Ω . Our assumption is wrong so $x^0 \notin \Omega$ that is $x^0 \in \partial\Omega$ and $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Pf case II $\Delta u \geq 0$ in Ω $x \in \mathbb{R}^n$

for $\varepsilon > 0$ consider $v = u + \varepsilon(x_1^2 + x_2^2 + \dots + x_n^2) = u + \varepsilon|x|^2$

$$\Delta v = \Delta u + 2\varepsilon n > 0$$

Hence, for v , $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$



$\forall x \in \Omega$

$$u(x) + \varepsilon|x|^2 \leq \max_{\partial\Omega} (u + \varepsilon|x|^2)$$

$$u(x) \leq \max_{\partial\Omega} (u + \varepsilon|x|^2)$$

Ω is bdd $\exists M > 0$ st $|x|^2 < M$ $\forall x \in \Omega$

$$\max_{\partial\Omega} (u + \varepsilon|x|^2) \leq \max_{\partial\Omega} u + \max_{\partial\Omega} \varepsilon|x|^2 \leq \max_{\partial\Omega} u + \varepsilon \cdot M$$

$$\forall x \in \Omega \quad u(x) \leq \max_{\partial\Omega} u + \varepsilon \cdot M$$

we can take $\varepsilon > 0$ $u(x) \leq \max_{\partial\Omega} u$ $\forall x \in \Omega$ so $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Corollary 1 let Ω be a bdd domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$

if $\Delta u \leq 0$ in Ω then $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

Pf apply weak max principle to $-u$

Corollary 2 let Ω be bdd domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$

if $\Delta u = 0$ in Ω then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ AND $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

Hence, $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$

Thm Uniqueness for Dirichlet Bdry Cond The Dirichlet bdry value problem for

Laplace (Poisson) eqn has at most 1 soln

Pf let Ω be bdd domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Consider

$$\Delta u = F(x) \quad x \in \Omega$$

$$u|_{\partial\Omega} = f(x) \quad x \in \partial\Omega$$

let u_1, u_2 be two solns, then $v = u_1 - u_2$ satisfies $\Delta v = 0$ in Ω
 $v|_{\partial\Omega} = 0$

By corollary 2 $\max_{\bar{\Omega}} |v| = \max_{\partial\Omega} |v|$

$$v|_{\partial\Omega} = 0 \Rightarrow \max_{\bar{\Omega}} |v| = 0 \Rightarrow |v| = 0 \Rightarrow v = 0 \quad \forall x \in \Omega \Rightarrow u_1 = u_2$$

15 Mays Persenbe

Recall we proved weak max principle for Laplace Operator and uniqueness of soln for Laplace (Poisson) eqn with Dirichlet boundary cond.

Stability Thm let Ω be a bdd domain in $\mathbb{R}^2$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$

suppose u satisfies

$$\Delta u = F(x) \quad x \in \Omega$$

$$u = f(x) \quad x \in \Omega$$

if $|F(x)| \leq A$ in Ω , $|f(x)| \leq B$ on Ω only

Pf Ω is bdd so $\exists a_1, a_2 \in \mathbb{R}$ st $\forall x = (x_1, \dots, x_n) \in \Omega$

we have $a_1 < x < a_2$. Set $v = A(e^{a_2 - a_1} - e^{x_1 - a_1}) + B$

$$\Delta v = -A e^{x_1 - a_1} \leq 0$$

$$v|_{\partial\Omega} = A(e^{a_2 - a_1} - e^{x_1 - a_1}) + B \quad \forall x = (x_1, \dots, x_n) \in \partial\Omega$$

$$x_1 \leq a_2 \Rightarrow e^{a_2 - a_1} \geq e^{x_1 - a_1} \Rightarrow e^{a_2 - a_1} - e^{x_1 - a_1} \geq 0 \quad \text{so } v|_{\partial\Omega} \geq 0$$

Now consider $w = u - v$

$$\Delta w = \Delta u - \Delta v = F + A e^{x_1 - a_1}$$

$$e^{x_1 - a_1} > 1 \quad \text{since } x_1 - a_1 \geq 0 \text{ in } \Omega \quad \Rightarrow \quad |F| \leq A$$

$$\text{so } \Delta w = F + A e^{x_1 - a_1} \geq 0 \text{ in } \Omega$$

By the weak max principle $\max_{\bar{\Omega}} w = \max_{\partial\Omega} w$

$$\text{Now, } w|_{\partial\Omega} = (u - v)|_{\partial\Omega}$$

$$= f - (A(e^{a_2 - a_1} - e^{x_1 - a_1}) + B) \quad (e^{a_2 - a_1} - e^{x_1 - a_1}) \geq 0$$

$$A \geq 0 \quad \Rightarrow \quad |f| \leq B \quad \Rightarrow \quad w|_{\partial\Omega} \leq 0$$

$$\Rightarrow \max_{\bar{\Omega}} w = \max_{\partial\Omega} w \leq 0$$

$$\forall x \in \Omega \quad w(x) \leq 0$$

$$u(x) - A(e^{a_2 - a_1} - e^{x_1 - a_1}) + B \leq 0 \quad \forall x \in \Omega$$

$$u(x) \leq A e^{a_2 - a_1} + B - \underbrace{A e^{x_1 - a_1}}_{\geq 0}$$

$$u(x) \leq A e^{a_2 - a_1} + B, \quad \text{set } C = e^{a_2 - a_1} : \text{const. depending only on } \Omega$$

$$u \leq CA + B \quad \forall x \in \Omega$$

in the same way considering $-u$ we can show $-u \leq CA + B \quad \forall x \in \Omega$

$$\text{hence, } |u| \leq CA + B$$

Solution of Dirichlet (Neumann) problem for Simple Domains

ex (Rectangle) consider $R \subset \mathbb{R}^2$ $R: 0 \leq x \leq a$ $0 \leq y \leq b$

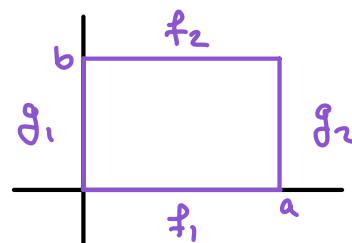
$$u_{xx} + u_{yy} = 0 \quad (x, y) \in R$$

$$u(x, 0) = f_1(x)$$

$$u(x, b) = f_2(x)$$

$$u(0, y) = g_1(y)$$

$$u(a, y) = g_2(y)$$



We can write $u = u_1 + u_2 + u_3 + u_4$

where $\Delta u_i = 0$ in R $i = 1, 2, 3, 4$ and u_i satisfies following bound cond

$$u_1: u_1(x, 0) = f_1, \quad u_1(x, b) = 0, \quad u_1(0, y) = 0, \quad u_1(a, y) = 0$$

$$u_2: u_2(x, 0) = 0, \quad u_2(x, b) = f_2, \quad u_2(0, y) = 0, \quad u_2(a, y) = 0$$

$$u_3: u_3(x, 0) = 0, \quad u_3(x, b) = 0, \quad u_3(0, y) = g_1, \quad u_3(a, y) = 0$$

$$u_4: u_4(x, 0) = 0, \quad u_4(x, b) = 0, \quad u_4(0, y) = 0, \quad u_4(a, y) = g_2$$

problems for u_i are same. we consider problem for u_1

$$\text{Set } u_1 = w$$

$$w_{xx} + w_{yy} = 0 \quad (x, y) \in R$$

$$w(x, 0) = f_1(x) \quad w(x, b) = 0$$

$$w(0, y) = 0 \quad w(a, y) = 0$$

$$\text{Set } w = X(x) \cdot Y(y)$$

$$X'' + \lambda X + X Y'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

rhs $\neq$ only
lhs $\times$ only
hence constant

$$w(x, b) = 0 : X(x) Y(b) = 0 \Rightarrow Y(b) = 0$$

$$w(0, y) = 0 : X(0) Y(y) = 0 \Rightarrow X(0) = 0$$

$$w(a, y) = 0 : X(a) Y(y) = 0 \Rightarrow X(a) = 0$$

$$\text{first consider } X'' + \lambda X = 0 \quad X(0) = 0 \quad X(a) = 0$$

for this boundary value problem we know eigenvalues $\lambda_n = \left(\frac{n\pi}{a}\right)^2$

and corresponding eigen functions $\bar{X}_n = \sin\left(\frac{n\pi}{a}x\right)$

Y_n ?

$$Y_n'' - \lambda_n Y_n = 0, \quad Y_n(b) = 0$$

$$Y_n = c_1 e^{\sqrt{\lambda_n} y} + c_2 e^{-\sqrt{\lambda_n} y} \quad (r^2 - \lambda_n = 0 \quad r = \sqrt{\lambda_n})$$

$$Y_n(b) = 0 \Rightarrow c_1 e^{\sqrt{\lambda_n} b} + c_2 e^{-\sqrt{\lambda_n} b} = 0$$

$$\text{we set } c_1 = c_n e^{-\sqrt{\lambda_n} b}$$

$$c_2 = -c_n e^{\sqrt{\lambda_n} b}$$

$$Y_n = c_n (e^{\sqrt{\lambda_n} y} \cdot e^{-\sqrt{\lambda_n} b} - e^{-\sqrt{\lambda_n} y} e^{\sqrt{\lambda_n} b})$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$Y_n = 2c_n \left(\frac{e^{\sqrt{\lambda_n}(y-b)} - e^{-\sqrt{\lambda_n}(y-b)}}{2} \right) = c_n \sinh(\sqrt{\lambda_n}(y-b))$$

$$\text{we have soln } w_n = \tilde{X}_n \cdot Y_n$$

$$w_n = c_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}(y-b)\right)$$

$$\text{we can construct a formal solution } w = \sum_{n=1}^{\infty} w_n$$

$$w = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}(y-b)\right)$$

$$w \text{ satisfies the } w(0,y) = 0, \quad w(a,y) = 0, \quad w(x,b) = 0$$

$$\text{the last cond } w(x,0) = f(x)$$

$$f(x) = w(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(-\frac{n\pi b}{a}\right)$$

$$\text{we have sine series for } f(x) \text{ on } [0,a]$$

$$-c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$c_n = -\frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx$$

ex Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \quad 0 < x < \pi \quad 0 < y < \pi$$

$$u(x, 0) = 0 \quad u(x, \pi) = \sin 3x + 2 \sin 5x$$

$$u(0, y) = 1 \quad u(\pi, y) = 0$$

Set $u = w + v$

$$w: \quad w_{xx} + w_{yy} = 0 \quad 0 < x < \pi \quad 0 < y < \pi$$

$$w(x, 0) = 0 \quad w(x, \pi) = \sin 3x + 2 \sin 5x$$

$$w(0, y) = 0 \quad w(\pi, y) = 0$$

$$v: \quad v_{xx} + v_{yy} = 0 \quad 0 < x < \pi \quad 0 < y < \pi$$

$$v(x, 0) = 0 \quad v(x, \pi) = 0$$

$$v(0, y) = 1 \quad v(\pi, y) = 0$$

Now we solve v , because the boundary values in y are 0 at $y = 0, \pi$ we use sine expansion in the y -direction and exponential (hyperbolic sine) decay in x .

Fourier sine series of the constant "1"

on $0 < y < \pi$

$$1 = \sum_{n=1}^{\infty} b_n \sin(ny) \quad , \quad b_n = \frac{2}{\pi} \int_0^{\pi} \sin(ny) dy = \frac{2}{\pi n} (1 - (-1)^n) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

now separation of var. $v(x, y) = X \cdot Y$

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

solving Y -part first because boundary data are simple

we know $v(x, 0) = v(x, \pi) = 0$ that forces $Y(0) = Y(\pi) = 0$

$$\text{and we have } Y'' + \lambda Y = 0$$

the only non trivial solns that vanish at both ends of $(0, \pi)$ are

$$Y_n = \sin(ny) \quad , \quad \lambda = n^2 \quad , \quad n = 1, 2, \dots \quad \text{so the eigenvalues are}$$

$$\lambda = n^2$$

solve the x-part with same eigenvalues $\lambda = n^2$

$$X'' - n^2 X = 0 \Rightarrow X(x) = A_n e^{nx} + B_n e^{-nx}$$

its usually easier to write those exponentials as hyperbolic func.

$$X(x) = C_n \sinh(nx) + D_n \cosh(nx)$$

for v , we need $v(0, y) = 1$, $v(\pi, y) = 0$

make the right edge $x = \pi$ value zero automatically

trick: $X_n(x) = \sinh(n(\pi-x)) = \sinh(nx) \cosh(n\pi) - \cosh(nx) \sinh(n\pi)$

because $X_n(\pi) = \sinh(n(\pi-\pi)) = 0$, the $x = \pi$ boundary satisfied for free

evaluate X_n at $x=0$

left edge $X_n(0) = \sinh(n\pi)$. Hence if a mode comes with coeff a_n

its contribution to $v(0, y)$ is $a_n X_n(0) Y_n(y) = a_n \sinh(n\pi) \cdot \sin(ny)$

match the left edge cond $v(0, y) = 1$

we now need a sine series of constant 1

$$1 = \sum_{n=1}^{\infty} b_n \sin(ny), \quad b_n = \frac{2}{\pi} \int_0^{\pi} \sin(ny) dy = \frac{2}{\pi n} (1 - (-1)^n) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

to reproduce that same series we must choose a_n s.t

$$a_n \sinh(n\pi) = b_n \Rightarrow a_n = \frac{b_n}{\sinh(n\pi)}$$

write the full series for v , put all together

$$v(x, y) = \sum_{n \text{ odd}} \frac{4}{\pi n} \frac{\sinh(n(\pi-x))}{\sinh(n\pi)} \sin(ny)$$

now we solve w

$$w_{xx} + w_{yy} = 0$$

$$w(x, 0) = 0$$

$$w(x, \pi) = g(x) = \sin 3x + 2\sin 5x$$

$$w(0, y) = 0$$

$$w(\pi, y) = 0$$

so, w vanishes on the bottom and on both vertical edges, but it is prescribed on the top.

Separate the var. $w(x, y) = X \cdot Y \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

solve x first because the horizontal edges are zero.

we need $w(0, y) = w(\pi, y) = 0$ so, $X(0) = X(\pi) = 0$

that forces X to be a sine eigenfunction $X_n(x) = \sin(nx)$, $\lambda = n^2$, $n=1, 2, \dots$

solve y with $\lambda = n^2$

$$Y'' - n^2 Y = 0 \Rightarrow Y_n(y) = A_n e^{ny} + B_n e^{-ny} = C_n \sinh(ny) + D_n \cosh(ny)$$

impose the bottom boundary $y=0$: because the $w(x, 0) = 0$ ^{need} $\Rightarrow Y_n(0) = 0$

that kills the cosh term and leaves $Y_n(y) = \sinh(ny)$

$$w_n(x, y) = c_n \sin(nx) \sinh(ny) \quad n=1, 2, \dots$$

normalise the modes to match the top edge

at $y=\pi$ we want $w(x, \pi) = g(x)$. introduce normalising factor $\sinh(n\pi)$

$$w(x, y) = \sum_{n=1}^{\infty} c_n \sin(nx) \frac{\sinh(ny)}{\sinh(n\pi)} \quad \text{Now } w(x, \pi) = \sum c_n \sin(nx) = g(x)$$

Fourier sine coeff of the top data $g(x)$

for $0 < x < \pi$ $c_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx$. Because $g(x) = \sin 3x + 2\sin 5x$ is already a sine combination the integrals are trivial

$$c_3 = \frac{2}{\pi} \int_0^{\pi} \sin^2 3x dx = 1$$

$$c_5 = \frac{2}{\pi} \int_0^{\pi} 2\sin^2 5x dx = 2 \Rightarrow w(x, y) = \sin 3x \frac{\sinh(3y)}{\sinh(3\pi)} + 2\sin 5x \frac{\sinh(5y)}{\sinh(5\pi)}$$

all other c_n 's are zero

$$u = v + w \quad \square$$

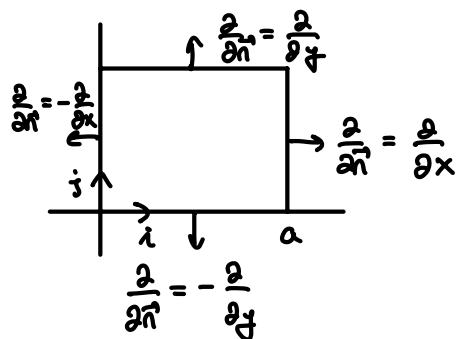
20 May 15 Salu

Neumann Problem for Laplace Eqn in a rectangle $\mathcal{R}: 0 < x < a, 0 < y < b$

$$u_{xx} + u_{yy} = 0 \quad (x, y) \in \mathcal{R}$$

$$\frac{\partial u}{\partial \vec{n}} = h(x, y) \quad (x, y) \in \mathcal{R}$$

$\frac{\partial}{\partial \vec{n}}$: directional derivative in the direction $\vec{n}$ unit normal to $\partial \mathcal{R}$



$\partial \mathcal{R}$ consist of 4 parts. As for Dirichlet problem

it is enough considering the case when $h(x, y)$ is

non-zero exactly on 1 part

Remark Soln of Neumann Problem is determined up to a constant.

$$\text{Soln exists iff } \int_{\partial \mathcal{R}} \frac{\partial u}{\partial \vec{n}} ds = \int_{\partial \mathcal{R}} h ds = 0 \quad [\text{pft later}]$$

ex Solve the Neumann problem

$$\Delta u = 0 \quad 0 < x < \pi \quad 0 < y < \pi$$

$$u_y(x, 0) = x - \frac{\pi}{2} \quad u_y(x, \pi) = 0$$

$$u_x(0, y) = 0 \quad u_x(\pi, y) = 0$$

$$\text{note } \int_{\partial \mathcal{R}} \frac{\partial u}{\partial \vec{n}} = 0 = \int_0^\pi x - \frac{\pi}{2} dx + 0 + 0 + 0 = 0$$

$$\text{let } u = X \cdot Y \quad \text{subst.} \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \quad \Rightarrow \quad \begin{aligned} X'' + \lambda X &= 0 \\ Y'' - \lambda Y &= 0 \end{aligned}$$

boundary cond.

$$u_y(x, \pi) = X \cdot Y'(\pi) = 0 \quad \Rightarrow \quad Y'(\pi) = 0$$

$$u_x(0, y) = X'(0) Y(y) = 0 \quad \Rightarrow \quad X'(0) = 0$$

$$u_x(\pi, y) = X'(\pi) Y(y) = 0 \quad \Rightarrow \quad X'(\pi) = 0$$

for X we have

$$X'' + \lambda X = 0 \quad X'(0) = 0, \quad X'(\pi) = 0$$

$$\boxed{\lambda = 0}: \quad X'' = 0 \quad \Rightarrow \quad X = A + Bx$$

$$\boxed{\lambda > 0} \text{ write } \lambda = n^2$$

$$X'' + n^2 X = 0 \quad \Rightarrow \quad X_n(x) = \cos(nx)$$

with eigenvalues $\lambda_n = n^2$

(all eigenfunction orthogonal in $L^2(0, \pi)$)

Neumann cond forces $B = 0$, $X_0(x) = A = 1$
take $A = 1$

for Y we have $Y'' - \lambda Y = 0$ $Y'(\pi) = 0$

$$\lambda_0 = 0 \ (n=0) : Y'' = 0 \Rightarrow Y_0 = c_1 y + c_2$$

$$Y'_0 = c_1 = 0 \Rightarrow Y_0 = c_2$$

$$\lambda_n = n^2 : Y_n'' - n^2 Y_n = 0, \quad Y_n'(\pi) = 0$$

$$r^2 - n^2 = 0 \\ (r-n)(r+n) = 0 \Rightarrow Y_n(y) = c_1 e^{ny} + c_2 e^{-ny}$$

$$Y_n'(y) = n c_1 e^{ny} - n c_2 e^{-ny} \Big|_{y=\pi} = n c_1 e^{n\pi} - n c_2 e^{-n\pi} = 0$$

$$c_1 = \frac{1}{2} c_n e^{-n\pi}$$

$$c_2 = \frac{1}{2} c_n e^{n\pi}$$

$$Y_n(y) = \frac{1}{2} c_n e^{-n\pi} e^{ny} + \frac{1}{2} c_n e^{n\pi} e^{-ny}$$

$$= c_n \frac{e^{n(y-\pi)} + e^{-n(y-\pi)}}{2} = c_n \cosh(n(y-\pi))$$

$$u_n(x, y) = X_n Y_n = c_n \cos(nx) \cdot \cosh(n(y-\pi))$$

we have formal soln $u(x, y) = c_0 \cdot \frac{1}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) \cosh(n(y-\pi))$

now we need to match it with $u_y(x, 0) = x - \pi/2$

$$u_y(x, y) = \sum_{n=1}^{\infty} c_n \cdot n \cdot \cos(nx) \sinh(n(y-\pi))$$

$$u_y(x, 0) = \sum_{n=1}^{\infty} c_n n \sinh(-n\pi) \cos(nx) = x - \frac{\pi}{2}$$

we have to expand $x - \pi/2$ into cosine series without 1st term. it's possible if the first term for $x - \pi/2$ is zero

cosine series of $x - \frac{\pi}{2} = \sum_{n=1}^{\infty} b_n \cos(nx)$ [note: $c_0 = \frac{2}{\pi} \int_0^{\pi} x - \frac{\pi}{2} dx = 0$ good ✓]

$$b_n = \frac{2}{\pi} \int_0^{\pi} (x - \pi/2) \cos(nx) dx = \frac{2((-1)^n - 1)}{\pi n^2} = \begin{cases} -\frac{4}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

match terms: $-c_n n \sinh(n\pi) = b_n \longrightarrow c_n = \begin{cases} \frac{4}{\pi n^3 \sinh(n\pi)} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$$u(x, y) = c_0 + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4}{\pi n^3 \sinh(n\pi)} \cos(nx) \cosh(n(y-\pi))$$

Dirichlet and Neumann problems in a disk

$$D: \{(x, y) : x^2 + y^2 < a^2\} \quad a > 0$$

$$\partial D = \{(x, y) : x^2 + y^2 = a^2\}$$

consider $\Delta u = 0$

$$u|_{\partial D} = f \quad \left(\frac{\partial}{\partial \vec{n}} u|_{\partial D} = f \right)$$

change of variables, we consider polar coordinates $x = r \cos \theta$, $y = r \sin \theta$

$$D: \{(r, \theta) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi\}$$

$$\partial D: \{(r, \theta) : r = a, 0 \leq \theta \leq 2\pi\}$$

Laplacian in Polar Coordinates

$$u(r, \theta) = u(x(r, \theta), y(r, \theta))$$

$$\Delta u = u_{xx} + u_{yy} = ?$$

$$u_r = u_x \cdot x_r + u_y \cdot y_r = u_x \cos \theta + u_y \sin \theta$$

$$u_\theta = u_x x_\theta + u_y y_\theta = u_x (-r \sin \theta) + u_y (r \cos \theta) = -u_x r \sin \theta + u_y r \cos \theta$$

$$u_{rr} = u_{xx} \cos^2 \theta + u_{xy} \cos \theta \sin \theta + u_{yx} \sin \theta \cos \theta + u_{yy} \sin^2 \theta$$

$$u_{rr} = \cos^2 \theta u_{xx} + 2 \cos \theta \sin \theta u_{xy} + \sin^2 \theta u_{yy}$$

$$u_{\theta\theta} = u_{xx} r^2 \sin^2 \theta - u_{xy} r^2 \sin \theta \cos \theta - u_x r \cos \theta + u_{yx} r \cos \theta (-r \sin \theta) + u_{yy} r^2 \cos^2 \theta - u_y r \sin \theta$$

$$u_{\theta\theta} = r^2 \sin^2 \theta u_{xx} - 2r^2 \sin \theta \cos \theta u_{xy} + r^2 \cos^2 \theta u_{yy} - r \cos \theta u_x - r \sin \theta u_y$$

$$r^2 u_{rr} + u_{\theta\theta} = r^2 u_{xx} + r^2 u_{yy} - r \cos \theta u_x - r \sin \theta u_y$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \underbrace{\frac{1}{r} (\cos \theta u_x + \sin \theta u_y)}_{u_r}$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

Separation of var. of Laplacian in Polar Coord.

$u(r, \theta) = R(r) \cdot \phi(\theta)$ then substit u into the eqn we get

$$R''\phi + \frac{1}{r^2} R\phi'' + \frac{1}{r} R'\phi = 0$$

$$\frac{R'' + \frac{1}{r} R'}{\frac{1}{r^2} R} = - \frac{\phi''}{\phi} = \lambda \quad \begin{array}{l} \text{lhs fn of } r \\ \text{rhs fn of } \theta \end{array} \Rightarrow \text{hence constant}$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\begin{cases} r^2 R'' + r R' - \lambda R = 0 \\ \phi'' + \lambda \phi = 0 \end{cases}$$

ϕ is 2π periodic $\phi(\theta) = \phi(\theta + 2\pi)$

we consider $\phi'' + \lambda \phi = 0$
eigenvalue problem on the space of 2π -periodic

$\lambda < 0$ $\lambda = -\mu^2$

char eqn $r^2 - \mu^2 = 0 \Rightarrow r = \pm \mu$

$\phi = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}$ no non-trivial 2π periodic soln

$\lambda = 0$ $\phi'' = 0 \Rightarrow \phi = c_1 \theta + c_2$: we have non-trivial 2π periodic soln
 $\phi = c_2$ ($\lambda_0 = 0$ eigenvalue $\phi_0 = A_0$ is eigenfn)

$\lambda > 0$ $\lambda = \mu^2$

char eqn $r^2 + \mu^2 = 0 \Rightarrow r = \pm i\mu$

$$\phi = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta)$$

Remark: $\cos(ax)$
 $\sin(ax)$ has periods $\frac{2\pi n}{a}$

Hence we have ϕ 2π periodic $\mu = n$

$\lambda_n = n^2$ eigenfunctions with eigenvalues $\phi_n = A_n \cos(n\theta) + B_n \sin(n\theta)$

correspond. R_n

$\lambda_0 = 0$ $r R'' + R' = 0$

$\lambda_n = n^2$ $r R'' + r R' - n^2 R = 0$

Remark on eqn $x^2 y'' + axy' + by = 0$: Euler Eqn

we solve it by introducing new indep var t by $x = e^t$

in t -variable we get eqn with constant coefficients

$$\text{for, } x^2 y'' + axy' + by = 0$$

we consider charact. eqn

$$\zeta(\zeta-1) + a^2 \zeta + b = 0$$

if ζ_1, ζ_2 two real distinct roots

$$y = c_1 x^{\zeta_1} + c_2 x^{\zeta_2}$$

if $\zeta_1 = \zeta_2$ double root

$$y = c_1 x^{\zeta_1} + c_2 \ln x \cdot x^{\zeta_1}$$

Hence, we get

$$\lambda_0 = 0, \quad r^2 R'' + r R' = 0$$

$$\zeta(\zeta-1) + \zeta = 0 \Rightarrow \zeta = 0 \text{ double root}$$

$$R_0 = c_1 + c_2 \ln r$$

$$\lambda_n = n^2 \quad \zeta(\zeta-1) + \zeta - n^2 = 0 \Rightarrow \zeta^2 - n^2 = 0 \Rightarrow \zeta = \pm n$$

$$R_n = c_1 r^n + c_2 r^{-n}$$

22 May 15 Pers

Recall we considered Dirichlet problem for a disk

$$\Delta u = 0 \quad x^2 + y^2 < a$$

$$u = f \quad x^2 + y^2 = a$$

introduce polar coord. $x = r \cos \theta$, $y = r \sin \theta$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad 0 \leq r \leq a$$

$$u(a, \theta) = f$$

Assume u is 2π periodic $\neq$ wrt θ and f 2π periodic

separate the var $u = R(r) \phi(\theta)$

$$\begin{cases} R'' + \frac{1}{r} R' + \frac{\lambda}{r^2} R = 0 \\ \phi'' + \lambda \phi = 0 \end{cases} \quad \phi(\theta) = \phi(\theta + 2\pi)$$

Solving the eigenvalue problem

$$\phi'' + \lambda \phi = 0 \quad \phi(\theta + 2\pi) = \phi(\theta)$$

we get $\lambda_0 = 0$, $\lambda_n = n^2$ with correspond. eigenfunction

$$\text{eigenfunction } \phi_0 = A_0 \quad \phi_n = A_n \cos(n\theta)$$

next, find corresp. R_n

$$\lambda_0 = 0 \quad R_0'' + \frac{1}{r} R_0' = 0 \Rightarrow R_0 = C_0 + D_0 \ln r$$

$$\lambda_n = n^2 \quad R_n'' + \frac{1}{r} R_n' - \frac{n^2}{r^2} R_n = 0$$

$$\Rightarrow r^2 R_n'' + r R_n' - n^2 R_n = 0 \Rightarrow R_n = C_n r^n + D_n r^{-n}$$

Continuing from where we left

we found particular soln

$$u_0(r, \theta) = A_0 (C_0 + D_0 \ln r)$$

$$u_n(r, \theta) = (A_n \cos(n\theta) + B_n \sin(n\theta)) (C_n r^n + D_n r^{-n})$$

recall $0 \leq r \leq a$ we need u_n to be cont on $[0, a]$ wrt r

$$r^{-n}, \ln r \text{ are not bounded } [0, a] \quad \lim_{r \rightarrow 0^+} \ln r \rightarrow -\infty \quad r^{-n} \xrightarrow{r \rightarrow 0^+} +\infty$$

$$u_0 = A_0 C_0 = \frac{\alpha_0}{2}$$

$$u_n = (A_n \cos(n\theta) + B_n \sin(n\theta)) C_n r^n = \alpha_n r^n \cos(n\theta) + \beta_n r^n \sin(n\theta)$$

u_0 and u_n are the solns of the Laplace Eqn

consider the formal series soln

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) , \quad u(r, \theta + 2\pi) = u(r, \theta)$$

now we satisfy $u(a, \theta) = f(\theta)$

$$f(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} a^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta))$$

using formula for the Fourier coeff for $[-\pi, \pi]$ ($[0, 2\pi]$)

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$a^n \alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta \Rightarrow \alpha_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$

$$a^n \beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta \Rightarrow \beta_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$$

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) : \text{ formal series soln}$$

OR

$$u(r, \theta) = \frac{\bar{\alpha}_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta)) \quad \text{where}$$

$$\bar{\alpha}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$\bar{\alpha}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$

$$\bar{\beta}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$$

it can be shown if f is 2π -periodic fn (f is defined on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$) with piecewise cont derivative f' , then the above series is a **classical soln** of the problem

ex Solve the problem

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

$$(u(1, \theta) = y^2) \quad u = y^2, \quad x^2 + y^2 = 1$$

$$x^2 + y^2 = r^2$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

in polar coord. $x = r \cos \theta$, $y = r \sin \theta$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(1, \theta) = r^2 \sin^2 \theta \Big|_{r=1} = \sin^2 \theta, \quad 0 \leq \theta \leq 2\pi \quad \text{OR} \quad -\pi \leq \theta \leq \pi$$

we know $a=1$

$$u = \frac{\bar{\alpha}_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{1}\right)^n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta))$$

$$\underline{r=1} \quad u(1, \theta) = \sin^2 \theta = \frac{\alpha_0}{2} = \frac{1 - \cos 2\theta}{2} = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

$$\begin{aligned} \bar{\alpha}_0 &= 1 & \bar{\alpha}_2 &= -\frac{1}{2} \\ \bar{\alpha}_n &= 0 & \bar{\beta}_n &= 0 \\ n \neq 2 & & \forall n & \end{aligned}$$

$$u(r, \theta) = \frac{1}{2} - \frac{1}{2} r^2 \cos(2\theta) = \frac{1}{2} - r^2 (\cos^2 \theta - \sin^2 \theta) = \frac{1}{2} - \frac{1}{2} (r \cos \theta)^2 + \frac{1}{2} (r \sin \theta)^2$$

another soln for above problem

$$= \frac{1}{2} - \frac{1}{2} x^2 + \frac{1}{2} y^2$$

$$u(r, \theta) = R \cdot \phi \Rightarrow \frac{r^2 R'' + r R'}{R} = - \frac{\phi''}{\phi} = \lambda$$

Solve 2 e-value problems

$$\phi'' + \lambda \phi = 0 \quad \text{AND} \quad r^2 R'' + r R' - \lambda R = 0$$

Angular e-functions (2π periodic)

$$\phi_0 = 1 \quad \phi_n^{\cos} = \cos(n\theta) \quad \phi_n^{\sin} = \sin(n\theta) \quad \text{with e-values} \quad \begin{aligned} \lambda_0 &= 0 \\ \lambda_n &= n^2 \end{aligned}$$

Radial e-functions

for each $\lambda = n^2$ the radial ODE is Euler

$$r^2 R'' + r R' - n^2 R = 0 \Rightarrow R(r) = A_n r^n + B_n r^{-n}$$

$$r=0 \text{ forces } B_n = 0 \quad n \geq 1$$

for $n=0$: $r^2 R'' + r R' = 0$ integrates to $R_0 = A_0 + B_0 \ln r$ ($\ln r$ is singular at 0 $\Rightarrow B_0 = 0$)

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

$$r=1: u(1, \theta) = \sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \quad \text{Thus } A_0 = \frac{1}{2}, \quad A_2 = -\frac{1}{2}, \quad A_n = 0, \quad B_n = 0$$

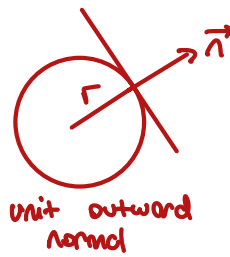
$$n \neq 0, 2 \quad \forall n$$

only the constant and $n=2$ survives $\frac{1}{2} - \frac{1}{2} r^2 \cos(2\theta) \quad \square$

ex Solve the problem

$$\Delta u = 0 \quad x^2 + y^2 < 4$$

$$\frac{\partial u}{\partial \vec{n}} = xy$$



$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial u}{\partial r}$$

polar coord: $x = r \cos \theta$, $y = r \sin \theta$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad 0 \leq r \leq 2 \quad -\pi \leq \theta \leq \pi$$

$$\left. \frac{\partial u}{\partial r} \right|_{r=2} = r^2 \sin \theta \cos \theta \Big|_{r=2} = 4 \sin \theta \cos \theta \quad -\pi \leq \theta \leq \pi$$

we know $a=2$

$$u = \frac{\bar{\alpha}_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta))$$

$$\left. \frac{\partial u}{\partial r} \right|_{r=2} = \sum_{n=1}^{\infty} n \left(\frac{r}{2}\right)^{n-1} (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta)) \Big|_{r=2} = \sum_{n=1}^{\infty} n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta))$$

||

$$4 \sin \theta \cos \theta = 2 \sin(2\theta)$$

$$2 \sin(2\theta) = \sum_{n=1}^{\infty} n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta))$$

$$\bar{\alpha}_n = 0 \quad n = 1, 2, \dots$$

$$2\bar{\beta}_2 = 2 \quad \bar{\beta}_1 = 0, \quad \bar{\beta}_3 = 0 \dots$$

$$\begin{aligned} u &= \frac{\bar{\alpha}_0}{2} + \left(\frac{r}{2}\right)^2 \sin(2\theta) = \frac{\bar{\alpha}_0}{2} + \frac{1}{4} r^2 \sin(2\theta) = \frac{\bar{\alpha}_0}{2} + \frac{1}{4} r^2 2 \sin \theta \cos \theta \\ &= \frac{\bar{\alpha}_0}{2} + \frac{1}{2} r^2 \sin \theta \cos \theta \end{aligned}$$

Return to x-y

$$u = \frac{\bar{\alpha}_0}{2} + \frac{1}{2} xy = c + \frac{1}{2} xy$$

Poisson Integral Formula

we consider $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad 0 \leq r \leq a^2 \quad -\pi \leq \theta \leq \pi$

$u(a, \theta) = f(\theta) \quad -\pi \leq \theta \leq \pi \quad f: 2\pi\text{-periodic}$

we know $u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\bar{\alpha}_n \cos(n\theta) + \bar{\beta}_n \sin(n\theta))$

$$\begin{aligned} u(r, \theta) &= \frac{1}{2} - \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta) \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta + \sin\theta \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta \\ &= \frac{1}{2} - \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi} f(\theta) \underbrace{(\cos(n\phi) \cos(n\theta) + \sin(n\theta) \sin(n\phi))}_{\cos(n(\phi-\theta))} d\theta \end{aligned}$$

formally

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\phi-\theta)) d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\phi-\theta)) \right] d\theta \end{aligned}$$

let us sum up $\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\phi-\theta))$

Recall $z = p(\cos \alpha + i \sin \alpha)$

$$z^n = p^n (\cos(n\alpha) + i \sin(n\alpha)) \Rightarrow \frac{z}{1-z} = \sum_{n=1}^{\infty} p^n \cos(n\alpha) + i \sum_{n=1}^{\infty} p^n \sin(n\alpha)$$

$$\operatorname{Re} \left(\frac{z}{1-z} \right) = \sum_{n=1}^{\infty} p^n \cos(n\alpha)$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} p^n \cos(n\alpha) = \frac{1}{2} + \operatorname{Re} \left(\frac{p \cos \alpha + i p \sin \alpha}{1 - (p \cos \alpha + i p \sin \alpha)} \right)$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} p^n \cos(n\alpha) = \frac{1-p^2}{2(1-2p \cos \alpha + p^2)}$$

$$u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \frac{1 - \left(\frac{r}{a}\right)^2}{2(1 - 2\left(\frac{r}{a}\right) \cos(\phi-\theta) + \left(\frac{r}{a}\right)^2)} d\theta$$

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{a^2 - r^2}{a^2 - 2ar \cos(\phi-\theta) + r^2} d\theta \quad : \text{Poisson Formula}$$

Poisson Kernel

One can prove

Thm let $f(\theta)$ be cont 2π periodic fn then

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{a^2 - r^2}{a^2 - 2ar \cos(\phi - \theta) + r^2} d\theta$$

$$u \in C^2(0 \leq r \leq a, -\pi \leq \theta \leq \pi) \cap C(0 \leq r \leq a, -\pi \leq \phi \leq \pi) \text{ and}$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad 0 \leq r < a \quad -\pi \leq \phi \leq \pi$$

$$u(a, \theta) = f(\theta) \quad -\pi \leq \phi \leq \pi$$

Remark solns of Laplace eqn $\Delta u = 0$ in D are called harmonic fns

u is harmonic iff $\Delta u = 0$ in D

Corollary (Mean Val Property) let u be harmonic in open domain D

and $B(a, (x_0, y_0)) \subset D$, $B(0, (x_0, y_0)) = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 \leq a^2\}$

$$u(x_0, y_0) = \frac{1}{2\pi a} \int_{\partial B(a, (x_0, y_0))} u(x, y) ds$$

27 May 18 Soln

Integral Representation of Soln

Green Identities

in what follow we consider $u: \mathbb{R}^3 \rightarrow \mathbb{R}$ and use notations

$$\text{grad } u = \nabla u = u_x i + u_y j + u_z k$$

$$F = F_1 i + F_2 j + F_3 k \quad (F_i: \mathbb{R}^3 \rightarrow \mathbb{R} \quad i=1,2,3)$$

$$\text{div } F = \nabla \cdot F = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\Delta u = \nabla \cdot (\nabla u) = u_{xx} + u_{yy} + u_{zz}$$

Divergence Thm let $D \subset \mathbb{R}^3$ be a regular bdd domain with piecewise smooth boundary and F be a smooth vector field, then

$$\iiint_D \text{div } \vec{F} dv = \iint_{\partial D} \vec{F} \cdot \vec{n} ds \quad : \quad \vec{n} \text{ is the outward normal}$$



Green's First Identity

let D be a domain where the Div Thm holds and $u, v \in C^2(D) \cap C^1(\bar{D})$

$$\text{Then } \nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \Delta u$$

By the Div Thm

$$\iiint_D \nabla \cdot (v \nabla u) dV = \iint_{\partial D} v \nabla u \cdot \vec{n} ds$$

$$\nabla u \cdot \vec{n} = \frac{\partial u}{\partial \vec{n}}$$

Green's First Identity

$$\iiint_D \nabla v \cdot \nabla u dV + \iiint_D v \Delta u dV = \iint_{\partial D} v \frac{\partial u}{\partial \vec{n}} ds$$

Some corollaries

set $v=1$

$$\iiint_D \Delta u dV = \iint_{\partial D} \frac{\partial u}{\partial \vec{n}} ds \quad (**)$$

set $v=u$

$$\iiint_D |\nabla u|^2 dV + \iiint_D u \Delta u dV = \iint_{\partial D} u \frac{\partial u}{\partial \vec{n}} ds \quad (***)$$

Neumann Problem

Lemma 1 (uniqueness for the Neumann Problem)

let $F \in C(D)$ and $f \in C(\partial D)$. if u_1 and u_2 are two soln of

$$\Delta u = F(x) \quad x \in D$$

$$\frac{\partial u}{\partial \vec{n}} = f \quad x \in \partial D$$

then $u_1 - u_2 = C$, $C \in \mathbb{R}$

Pf let $v = u_1 - u_2$ then $\Delta v = 0$ in D Then apply (***) to v

$$\frac{\partial v}{\partial \vec{n}} = 0 \quad \text{on } \partial D$$

$$\iiint_D |\nabla v|^2 dV + \iiint_D v \underbrace{\Delta v}_0 dV = \iint_{\partial D} v \underbrace{\frac{\partial v}{\partial \vec{n}}}_0 ds$$

Hence $\iiint_D |\nabla v|^2 = 0 \Rightarrow \nabla v = 0 \quad v_x = 0 \quad v_y = 0 \quad v_z = 0$

Thus $v = C$ that is $u_1 - u_2 = C \quad C \in \mathbb{R}$

Neumann Problem

Lemma 2 (Necessary Cond for Neumann Problem)

let $F \in C(D)$ and $f \in C(\partial D)$. If the problem $\Delta u = F$ in D
 $\frac{\partial u}{\partial \vec{n}} = f$ in ∂D

has a soln then $\iiint_D F dV = \iint_{\partial D} f ds$

pf apply $\iiint_D \underbrace{\Delta u}_F dV = \iint_{\partial D} \underbrace{\frac{\partial u}{\partial \vec{n}}}_f ds$ (**) to u

$$\iiint_D F dV = \iint_{\partial D} f ds$$

Dirichlet Problem

Lemma 3 (uniqueness of soln for Dirichlet problem)

let $F \in C(D)$ and $f \in C(\partial D)$ then the problem

$$\begin{array}{ll} \Delta u = F & \text{in } D \\ u = f & \text{on } \partial D \end{array} \quad \text{has at most 1 solution}$$

pf let u_1 and u_2 be two soln $v = u_1 - u_2$ satisfies

$$\begin{array}{ll} \Delta v = 0 & \text{in } D \\ v = 0 & \text{on } \partial D \end{array} \quad \text{By } \iiint_D |\nabla u|^2 dV + \iiint_D u \Delta u dV = \iint_{\partial D} u \frac{\partial u}{\partial \vec{n}} ds \quad (***)$$

$$\iiint_D |\nabla v|^2 dV + \iiint_D \underbrace{v \Delta v}_0 dV = \iint_{\partial D} \underbrace{v \frac{\partial v}{\partial \vec{n}}}_0 ds \quad (***)$$

$$\Rightarrow \iiint_D |\nabla v|^2 dV = 0$$

$$\Rightarrow |\nabla v|^2 = 0 \Rightarrow \nabla v = 0 \Rightarrow v = c \quad \text{note } v \in C^1(\bar{D}) \text{ and}$$

$$v|_{\partial D} = 0 \Rightarrow c = 0 \quad \text{Hence } v = 0 \Rightarrow u_1 = u_2 \text{ in } \bar{D}$$

Dirichlet Principle

for $w \in C^2(D) \cap C^1(\bar{D})$ we define

$$E(w) = \frac{1}{2} \iint_D |\nabla w|^2 dV \quad : \text{ energy functional}$$

Lemma $\forall w \in C^2(D) \cap C^1(\bar{D})$ that have fixed value of the energy functional is reached on the harmonic function

For any $w \in C^2(D) \cap C^1(\bar{D})$ st $w|_{\partial D} = h$

$$E(w) \geq E(u) \quad \text{where } u \in C^2(D) \cap C^1(\bar{D})$$

$$\text{and } \begin{array}{ll} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{array}$$

Pf consider $v = u - w$ or $w = u - v$ ($v|_{\partial D} = 0$) Then $\nabla w = \nabla u - \nabla v$

$$|\nabla w|^2 = \nabla w \cdot \nabla w = (\nabla u - \nabla v) \cdot (\nabla u - \nabla v)$$

$$= |\nabla u|^2 - 2 \nabla u \cdot \nabla v + |\nabla v|^2$$

$$E(w) = \frac{1}{2} \iiint_D |\nabla w|^2 dV$$

$$= \frac{1}{2} \iiint_D |\nabla u|^2 dV - \iiint_D \nabla u \cdot \nabla v dV + \frac{1}{2} \iiint_D |\nabla v|^2 dV$$

$$\text{By Green's 1st Identity: } \left(\iiint_D \nabla v \cdot \nabla u dV + \iiint_D v \Delta u dV = \iint_{\partial D} v \frac{\partial u}{\partial \vec{n}} ds \right)$$

$$\iiint_D \nabla u \cdot \nabla v dV = - \iiint_D v \underbrace{\Delta u}_0 dV + \iint_{\partial D} v \underbrace{\frac{\partial u}{\partial \vec{n}}}_0 ds = 0$$

$$E(w) = E(u) + \underbrace{E(v)}_{\geq 0}$$

$$\text{so } E(w) \geq E(u)$$

Green's Function

Fundamental Soln Note that the Laplace Operator is rotationally invariant

So we can look for rotationally invariant soln that is the function

u dependent only on $r = \sqrt{x^2 + y^2 + z^2}$. let us find such solns

we consider $\mathbb{R}^n$ $r = \sqrt{x^2 + y^2 + z^2}$, let us solve $\Delta u = 0$

$u(r)$

$$u_{x_1} = u_r \cdot r_{x_1} = u_r \frac{x_1}{\sqrt{x_1^2 + \dots + x_n^2}} = u_r \cdot \frac{x_1}{r}$$

$$\begin{aligned} u_{x_1 x_1} &= u_{rr} \cdot r_{x_1} \frac{x_1}{r} + u_r \left(\frac{1}{r} + x_1 \frac{-1}{r^2} r_{x_1} \right) \\ &= u_{rr} \frac{x_1}{r} \frac{x_1}{r} + u_r \left(\frac{1}{r} + x_1 \frac{-1}{r^2} \frac{x_1}{r} \right) \\ &= \frac{x_1^2}{r^2} u_{rr} + \frac{1}{r} u_r - \frac{x_1^2}{r^3} u_r \end{aligned}$$

and similarly $u_{x_i x_i} = \frac{x_i^2}{r^2} u_{rr} + \frac{1}{r} u_r - \frac{x_i^2}{r^3} u_r$

$$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_n x_n}$$

$$\Delta u = \frac{x_1^2 + \dots + x_n^2}{r^2} u_{rr} + \frac{n}{r} u_r - \frac{x_1^2 + \dots + x_n^2}{r^3} u_r$$

$$\Delta u = u_{rr} + \frac{n}{r} u_r - \frac{1}{r} u_r = u_{rr} + \frac{n-1}{r} u_r$$

$$\Delta u = 0 \Rightarrow u_{rr} + \frac{n-1}{r} u_r = 0$$

we can find u explicitly

$n=2$	$u = A + B \ln r$	$A, B \in \mathbb{R}$
$n \geq 3$	$u = A + B \frac{1}{r^{n-2}}$	

we take the non constant soln

Define the fundamental soln as $u = \begin{cases} \ln r & n=2 \\ \frac{1}{r^{n-2}} & n \geq 3 \end{cases}$

Remark for any $x_0 \in \mathbb{R}^n$ we also have soln of $\Delta u = 0$

in $\mathbb{R}^n \setminus \{x_0\}$

$$u = \begin{cases} u|x-x_0| & n=2 \\ \frac{1}{|x-x_0|^{n-2}} & n \geq 3 \end{cases}$$

Thm let $u \in C^2(D) \cap C^1(\partial D)$ $D \subset \mathbb{R}^n$ be harmonic in D ($\Delta u = 0$ in D)

$$\text{Then } u(x_0) = \frac{1}{4\pi} \iint_{\partial D} \left(-u(x) \cdot \frac{\partial}{\partial \vec{n}} \frac{1}{|x-x_0|} + \frac{1}{|x-x_0|} \frac{\partial}{\partial \vec{n}} u(x) \right) ds, \quad \forall x \in D, \quad u|_{\partial D} = f$$

29 May 18 Pers

Recall we found the fundamental soln

$$u_{\text{fun}}(x) = \begin{cases} \ln|x| & x \in \mathbb{R}^2 \\ \frac{1}{|x|^{n-2}} & x \in \mathbb{R}^n \end{cases}$$

for the Laplace operator
 $u_{\text{fun}}(x)$ on $\mathbb{R}^n \setminus \{x_0\}$ satisfy
if $\Delta u = 0$

we formulated thm

Thm let $u \in C^2(D) \cap C^1(\bar{D})$ $D \subset \mathbb{R}^n$ be harmonic in D ($\Delta u = 0$ in D)

$$\text{Then } u(x_0) = \frac{1}{4\pi} \iint_{\partial D} \left(-u(x) \cdot \frac{\partial}{\partial \vec{n}} \frac{1}{|x-x_0|} + \frac{1}{|x-x_0|} \frac{\partial}{\partial \vec{n}} u(x) \right) ds, \quad \forall x \in D, \quad u|_{\partial D} = f$$

Remark the proof is based on the Green's 2nd Formula

let $u, v \in C^2(D) \cap C^1(\bar{D})$ then

Green's 2nd Formula
$$\iiint_D (u \Delta v - v \Delta u) dA = \iint_{\partial D} \left(u \frac{\partial}{\partial \vec{n}} v - v \frac{\partial}{\partial \vec{n}} u \right) dS$$

$$\text{set } v = \frac{1}{4\pi} \frac{1}{|x-x_0|}$$

use Green's 2nd Formula

in $D_\varepsilon = D \setminus B(x_0, \varepsilon)$

easily obtained from

Green's 1st Formula

$$\iint_{\partial D} v \frac{\partial}{\partial \vec{n}} u \, dS = \iiint_D \nabla u \cdot \nabla v \, dA + \iint_D v \Delta u \, dA$$

$$\iint_{\partial D} u \frac{\partial}{\partial \vec{n}} v \, dS = \iiint_D \nabla u \cdot \nabla v \, dA + \iint_D u \Delta v \, dA$$

take difference we get the above formula

Green Function The Green Function for the Laplace operator in a domain D is a function $G(x, x_0)$

- 1) $\Delta_x G(x, x_0) = 0$ in $D \setminus \{x_0\}$
- 2) $G(x, x_0) = 0$ on ∂D
- 3) $G(x, x_0) + \frac{1}{4\pi} \frac{1}{|x - x_0|}$ is twice cont. diffable harmonic fn in D

Remark to find $G(x, x_0)$ we need to find harmonic fn $H(x, x_0)$ in D

st $H(x, x_0)|_{\partial D} = \frac{1}{4\pi} \frac{1}{|x - x_0|} \rightarrow G(x, x_0) = H(x, x_0) - \frac{1}{4\pi} \frac{1}{|x - x_0|}$

Thm let $G(x, x_0)$ be the Green Function of the Laplace operator for domain D

Then the soln of the Dirichlet problem

$$\begin{array}{l} \Delta u = 0 \quad \text{in } D \\ u = f(x) \quad \text{on } \partial D \end{array} \quad \text{is given by} \quad u(x_0) = \iint_{\partial D} f(x) \frac{\partial}{\partial \vec{n}} G(x, x_0) \, dS$$

pf we know $u(x_0) = \iint_{\partial D} \left(-u(x) \frac{\partial}{\partial \vec{n}} \frac{1}{4\pi} \frac{1}{|x - x_0|} + \frac{1}{4\pi} \frac{1}{|x - x_0|} \frac{\partial}{\partial \vec{n}} u \right) dS$

let $H(x, x_0) = G(x, x_0) + \frac{1}{4\pi} \frac{1}{|x - x_0|}$, we have $\Delta_x H = 0$, $H|_{\partial D} = \frac{1}{4\pi} \frac{1}{|x - x_0|}|_{\partial D}$

By Green's 2nd Formula applied to u, H

$$\iint_{\partial D} \left(u \frac{\partial}{\partial \vec{n}} H - H \frac{\partial}{\partial \vec{n}} u \right) dS \quad \text{Taking sum of}$$

$$u(x_0) = \iint_{\partial D} u \frac{\partial}{\partial \vec{n}} \left(\underbrace{H - \frac{1}{4\pi} \frac{1}{|x - x_0|}}_G \right) + \left(\underbrace{\frac{1}{4\pi} \frac{1}{|x - x_0|} - H}_0 \right) \frac{\partial}{\partial \vec{n}} u \, dS = u(x_0) = \iint_{\partial D} u \cdot \frac{\partial}{\partial \vec{n}} G \, dS$$

in the same way we can prove

Thm let $G(x, x_0)$ be the Green Function of the Laplace operator for a domain D

Then the soln for Dirichlet problem for the Poisson Eqn

$$\begin{aligned} \Delta u &= F(x) & \text{in } D \\ u &= f(x) & \text{on } \partial D \end{aligned} \quad \text{is given by}$$

$$u(x_0) = \iiint_D F(x) G(x, x_0) dV + \iint_{\partial D} f(x) \frac{\partial}{\partial n} G(x, x_0) dS$$

properties of Green's Function

1) G is symmetric ($G(x, x_0) = G(x_0, x)$ $x \neq x_0$ $x, x_0 \in D$)

2) G is unique

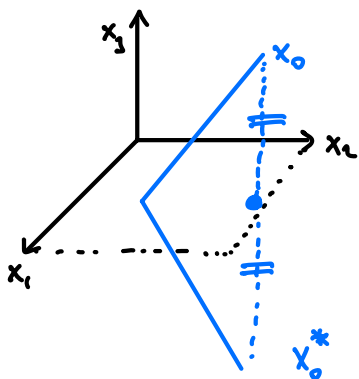
Green Function for Special Domains

Half Space $D = \{(x_1, x_2, x_3) \mid x_3 > 0\}$

note: D infinite domain

We assume that all fns and their derivative decreasing fast enough

as $|x| \rightarrow +\infty$ (all integrals are convergent)



$$\frac{1}{|x - x_0|} \quad \frac{1}{|x - x_0^*|}$$

for a pt $x_0 = (x_1^0, x_2^0, x_3^0)$

we consider symmetric (wrt boundary) point

$$x_0^* = (x_1^0, x_2^0, -x_3^0)$$

$$\text{and } H(x, x_0) = \frac{1}{4\pi |x - x_0^*|}$$

Then H is harmonic in D and $H|_{\partial D} = \frac{1}{4\pi |x - x_0|} \Big|_{\partial D}$

The Green Function

$$G(x, x_0) = \frac{1}{4\pi |x - x_0^*|} - \frac{1}{4\pi |x - x_0|} = \frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 + x_3^0)^2}} \\ - \frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2}}$$

for D : $\frac{\partial}{\partial \vec{n}} = -\frac{\partial}{\partial x_3}$

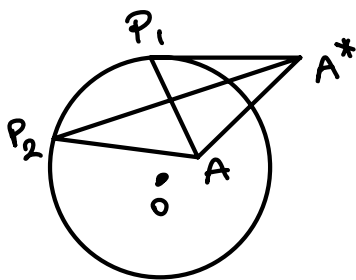
$$\frac{\partial}{\partial \vec{n}} G = -\frac{\partial}{\partial x_3} G = \frac{1}{2\pi} \frac{x_3^0}{|x - x_0|^3}$$

Hence, $\Delta u = 0$ in D

$u = f(x_1, x_2)$ on ∂D has solution

$$u(x_0) = \frac{x_3^0}{2\pi} \iint_{\mathbb{R}^2} \frac{f(x_1, x_2)}{((x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3^0)^2)^{\frac{3}{2}}} dx_1 dx_2$$

Sphere $D: \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq a^2\}$



for a pt $A \in D$ we define a symmetric pt A^*

by $\vec{OA^*} = \frac{a^2}{|OA|} \vec{OA}$

Lemma for any $p \in \partial D$ the ratio $\frac{|PA|}{|PA^*|}$ is constant, does not depend on P

$$\frac{|P_1 A^*|}{|P_1 A|} = \frac{|P_2 A^*|}{|P_2 A|} = \frac{|OA|}{a} \quad P_1, P_2 \in \partial D \quad (|OA| \cdot |OA^*| = a^2)$$

Now for $x_0 \in D$ we define $x_0^* = \frac{a^2}{|x_0|^2} x_0$. Since $x_0^* \notin D$

$H(x, x_0) = \frac{1}{4\pi} \frac{a}{|x_0|} \frac{1}{|x - x_0^*|}$. H is harmonic in D and $\forall x \in \partial D$

$$H(x, x_0) = \frac{1}{4\pi} \frac{1}{|x - x_0|}$$

So the Green Function for D

$$G(x, x_0) = \frac{1}{4\pi} \frac{a}{|x_0|} \frac{1}{|x - x_0^*|} - \frac{1}{4\pi} \frac{1}{|x - x_0|}$$

using defn of x_0^*

$$G(x, x_0) = \frac{1}{4\pi} \left[\left| \frac{|x_0|}{a} x - \frac{a}{|x_0|} x_0 \right|^{-1} - |x - x_0|^{-1} \right]$$

next we find $\frac{\partial}{\partial \vec{n}} G$. (Note $\frac{\partial}{\partial \vec{n}} = \nabla G \cdot \vec{n}$, $\vec{n} = \frac{\vec{x}}{a}$, $x \in \partial D$)